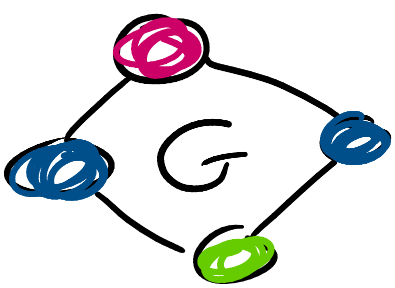
Properties of a k-chromatic graph

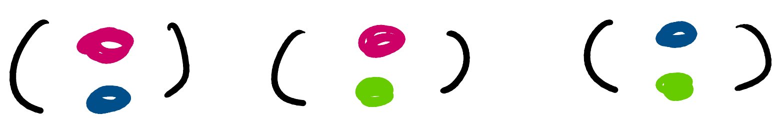
How small can a k-chromatic graph be?

- Consider all possible color pairs:
  → $\binom{k}{2}$ possible combinations
  → this is the minimum # edges on a k-chromatic graph

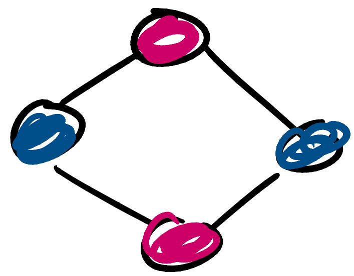
Why? Every combination must exist on a k-coloring of a k-chromatic graph's edges

→ Otherwise we can just combine colors





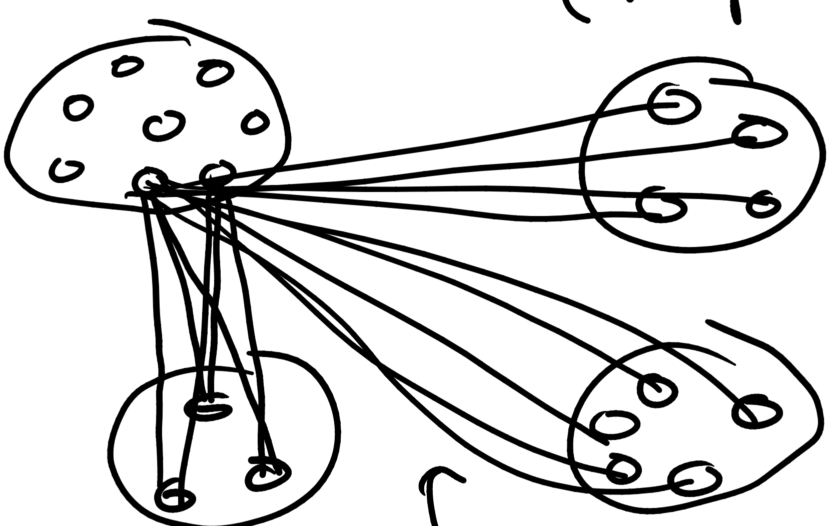
doesn't exist on G

combine: 

$\Rightarrow$ a k-chromatic graph has at least $\binom{k}{2}$ edges $\square$

How big can a k-chromatic graph be?

consider a multi-partite graph (k-partite)



To maximize edges: make it complete

repeat for all vertices

How do we further maximize

How do we further maximize for a given k, $|V(G)|$

→ set all partite sets equal in size $^+/-1$ (Turán graph)

Does it maximize $|E(G)|$?

- consider "unbalanced" multi-partite complete graph
- $\exists S_i\ S_j$ s.t. $|S_i| + 1 < |S_j|$
- consider moving $v \in S_j$ to S_i
- edges lost $= |S_i|$
- edges gained $= |S_j| - 1$

-> as $|S_j| > |S_i| + 1$

=> we have a net gain

=> Turán Graph is larget possible k-chromatic graph on $n = |V(G)|$ vertices $\square$

on $n = |V(G)|$ vertices $\square$

Color-critical graphs

G is color-critical if

$\forall u \in V(G) \rightarrow \chi(G-u) < \chi(G)$

$\forall e \in E(G) \rightarrow \chi(G-e) < \chi(G)$

For color-critical graph G

$\exists$ some k-coloring on G s.t.

$\forall v \in V(G)$ the color $c(v)$

appears nowhere else and

there is $k-1$ colors in $N(v)$

→ Consider $(k-1)$-coloring on $G-v$

→ add back in our v

→ if not all $k-1$ colors are

in that $N(v)$, we could

color v with one of those

that doesn't show up

$\rightarrow$ contradiction

Similarly: $\forall e = (u, v) \in E(G)$

$\rightarrow$ Every proper $(k-1)$-coloring of $G - e$ gives $C(u) = C(v)$

$\rightarrow$ If not, that'd equivalently be a $(k-1)$-coloring on G

Connectivity of k-color-critical graph G

Show: G is $(k-1)$-edge-connected

To do so, first show:

For G s.t. $\chi(G) > k$, let $\{X, Y\}$ be a partition of $V(G)$

If $G[X]$ and $G[Y]$ are both k-colorable $\rightarrow |[X, Y]| \geq k$

k-colorable $\rightarrow |[X,Y]| \geq k$

- Consider $X_1, X_2 \ldots X_k$ and $Y_1, Y_2 \ldots Y_k$ as independent sets per k-coloring

Show: if $|[X,Y]| < k$, $\exists$ (multiple) $X_i Y_j$ that we can combine to form a k-coloring on G

- Assume $|[X,Y]| < k$
- Construct H as a bigraph
  - All X_i, Y_j are a single vertex
  - Add edge (X_i, Y_j) if no edge $(x,y) \in E(G): x \in X_i, y \in Y_j$ exists

Note: H has more than $k(k-1)$ edges
$\rightarrow k^2$ possible, but cut $< k$

Note: m vertizes cover at most $k * m$ edges

$k * m$ edges

$\rightarrow E(H)$ is not covered by only $(k-1)$ vertices

Min cover $\geq k$

max match $\leq k$

min cover $=$ max match $= k$

If we combine ALL matched sets into a single color

$\rightarrow$ we a k-coloring on G

$\Rightarrow$ Contradiction

so $|[X,Y]| \geq k$

Bring it on home: 

Every k-critical graph is $(k-1)$-edge-connected

- Consider k-critical graph G

- Consider k-critical graph G
- $[X, Y]$ is some min cut

$\rightarrow G[X], G[Y]$ are $k-1$-colorable

Follows from prior proof:

$\Rightarrow$ edge cut must be at least $k-1$ in size $\square$

Min vertex coloring:

- NP-complete/NP-hard

takeaway: impossible to solve the min. coloring problem in the general case

Heuristics is the name of the game:

- Usually based on vertex order for greedy coloring
- degree based

→ degree based

→ Brelàz: go in order of which vertex currently has most colors in $N(v)$

BIG take away:

greedy coloring is Hugely dependent on processing order

QEWL $\boxed{\checkmark}$