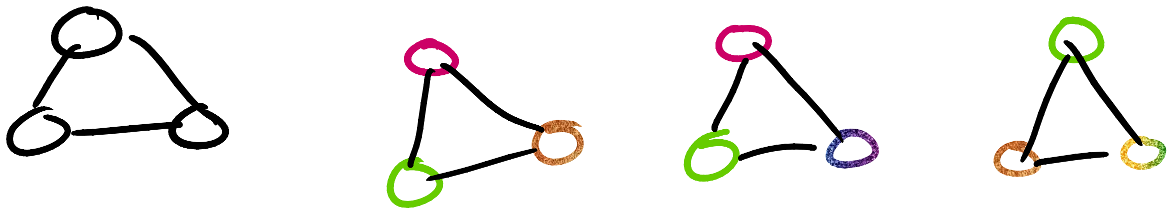
Lecture 20 - counting colors

Friday, April 2, 2021 1:28 PM

$\chi(G,k)$ = # of ways to color graph G with k colors

$\chi(G,k) = 0$ if $k < \chi(G)$

$k = 4 \to \{ \bullet\ \bullet\ \bullet\ \bullet \}$



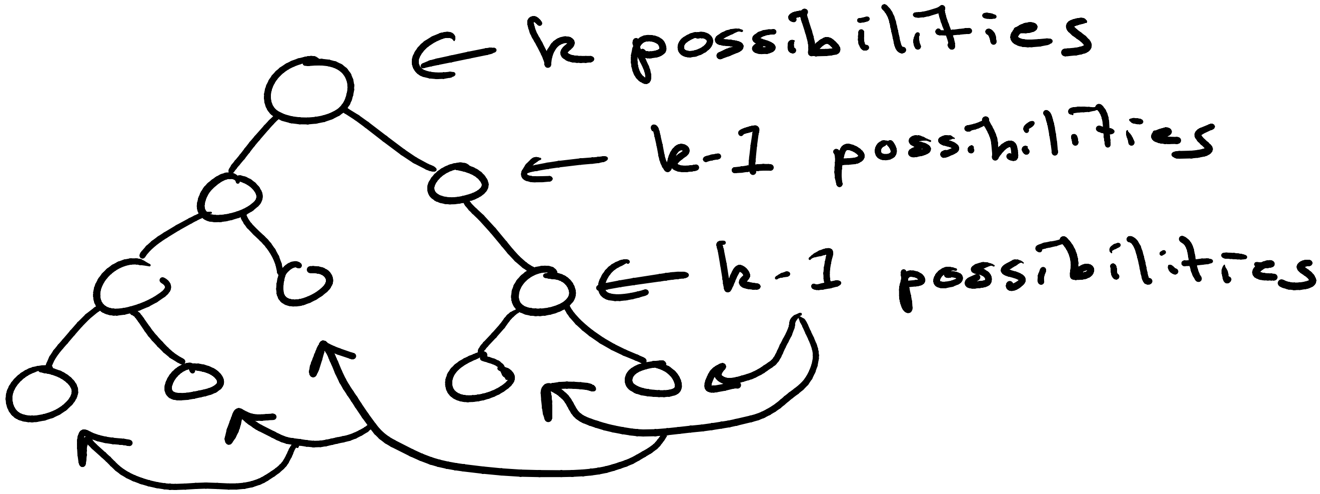
Consider clique K_n

- Color first vertex with any of k possible colors
- Color second vertex with any of $k-1$ possible

...

$$\chi(K_n, k) = k(k-1)(k-2)\cdots(k-n+1)$$

Let's talk treez



$$\chi(T, k) = k(k-1)^{n-1}$$

$\chi(G, k)$ = chromatic polynomial

General Form

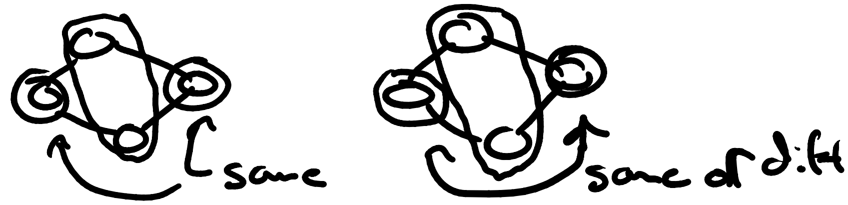
$$\chi(G, k) = \sum_{r=1}^{n} p_r(G)\, k_r$$

$P_r(G)$ = # of ways to partition G into r independent sets

k_r = # of ways to color these independent sets

$$k(k-1)\ldots(k-r+1)$$

$k(k-1)\dots(k-r+1)$

Consider C_4 

$p_1 = 0 \quad p_2 = 1 \quad p_3 = 2 \quad p_4 = 1$

$$\chi(C_4, k) = \sum_{r=1}^{n} p_r(G)\, k_r$$
$$= 0 + 1(k)(k-1)$$
$$+ 2(k)(k-1)(k-2)$$
$$+ 1(k)(k-1)(k-2)(k-3)$$

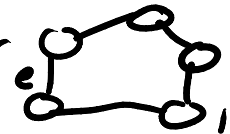
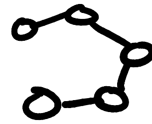
Fundamental Reduction Theorem

$$\chi(G, k) = \chi(G-e, k) - \chi(G \cdot e, k)$$
$$e = (u, v) \in V(G)$$

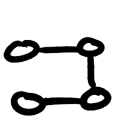
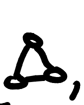
$\chi(G-e, k) =$ # of ways to color G with
$C(u) = C(v)$
$C(u) \neq C(v)$

$\chi(G \cdot e, k) =$ # of ways to

$\chi(G \cdot e, k)$ - # of ways to color G with $c(u) = c(v)$

$\chi($  $, k) = \chi($  $, k) - \chi($  $, k)$

↑ tree

$= k(k-1)^4 - (\chi($  $, k) - \chi($  $, k))$

$= k(k-1)^4 - k(k-1)^3 + k(k-1)(k-2)$

$\chi(G, 1) = 0 + 0 + 0 = 0$

$\chi(G, 2) = 2(1) - 2(1) + 0 = 0$

$\chi(G, 3) = 3(2^4) - 3(2^3) + 3(2)(1)$

$= 48 - 24 + 6$

$= 30$ ✓

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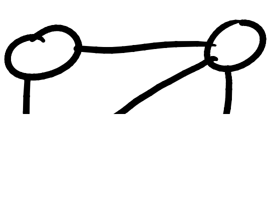
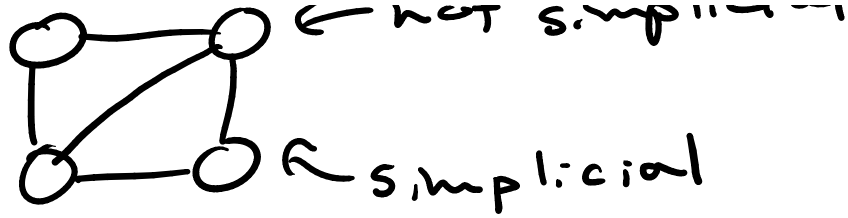
simplicial vertex = a vertex v where $N(v)$ forms a clique

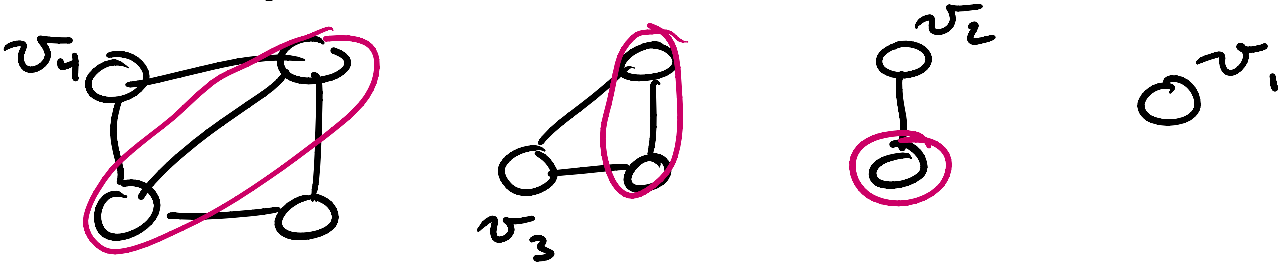
 ← not simplicial



Note: v forms a clique with $N(v)$

<u>Simplicial elimination ordering</u> (SEO)

an ordering $\{v_n, \ldots, v_1\}$ of all $v \in V(G)$ for deletion such that each vertex v_i is simplicial in the remaining graph induced on $\{v_i \ldots v_1\}$



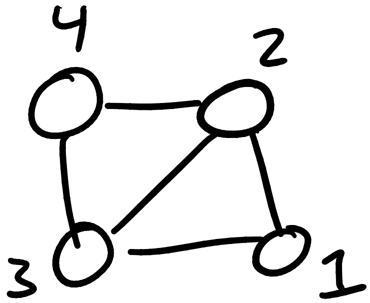
Note: we can construct a chromatic polynomial using this ordering

To get $\chi(G, k)$ using our SEO

- add $v_1 \ldots v_n$ to $G_i : G[\{v_1 \ldots v_i\}]$

$$\chi(G,k) = \prod_{i=1}^{n} (k - d'(v_i))$$

$$= k(k-1)(k-2)^2$$



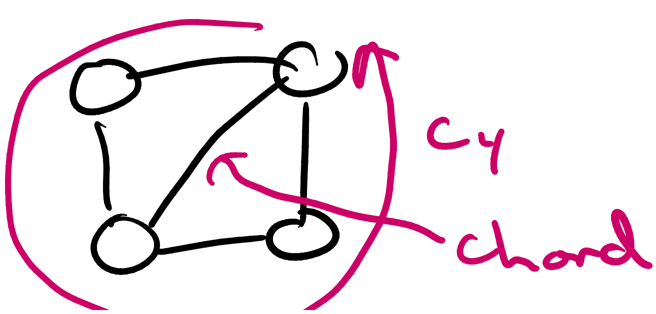
$d'(v_1) = 0$
$d'(v_2) = 1$
$d'(v_3) = 2$
$d'(v_4) = 2$

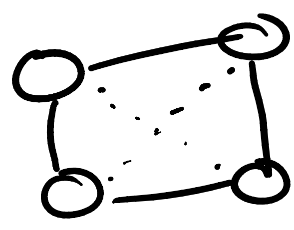
So: which graphs have a simplicial elimination ordering?

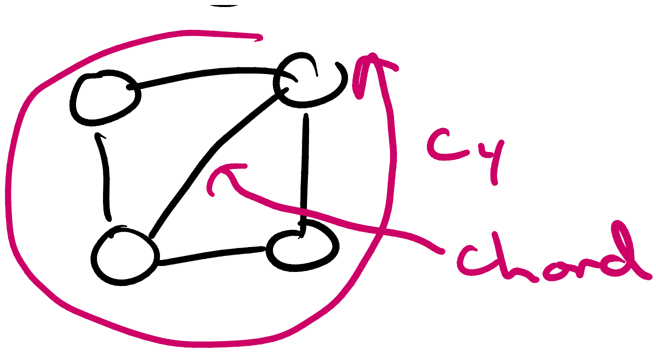
Answer: chordal graphs

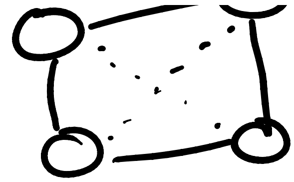
Chordal graphs: a simple graph that has no chordless cycle

Chord: an edge with endpoints on a cycle but is not on the cycle









Chordless cycles: a cycle of length at least 4 that has no chord

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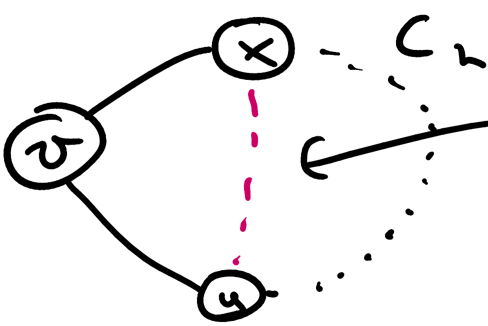
G has a SEO $\Leftrightarrow$ G is chordal

Has SEO $\Rightarrow$ G is chordal

- consider some $C_{n \geq 4} \in G$
- consider the first vertex v deleted on that cycle



note: if edge (x,y) didn't exist, v would not be simplicial ✓

G is chordal $\Rightarrow$ G has a SEO

show: every chordal graph has a

Show: every chordal graph has a simplicial vertex forthest from some vertex x

Note: deleting a vertex can't introduce a chordless cycle

Strong induction on $|V|$

Basis: $P(1)$ = single vertex is simplicial

$P(n > k)$: consider G, $|V(G)| = n$

- consider $x \in V(G)$

Case 1: $N(x) = \{V(G) - x\}$

→ $G - x$ is chordal

→ I.H. on $G - x$

→ Any simplicial vertex in $G - x$ is simplizial in G

$SEO(G) = \{x\} + SEO(G - x)$

Case 2: $N(x) \neq \{V(G) - x\}$

- define T = vertices maximum distance from x

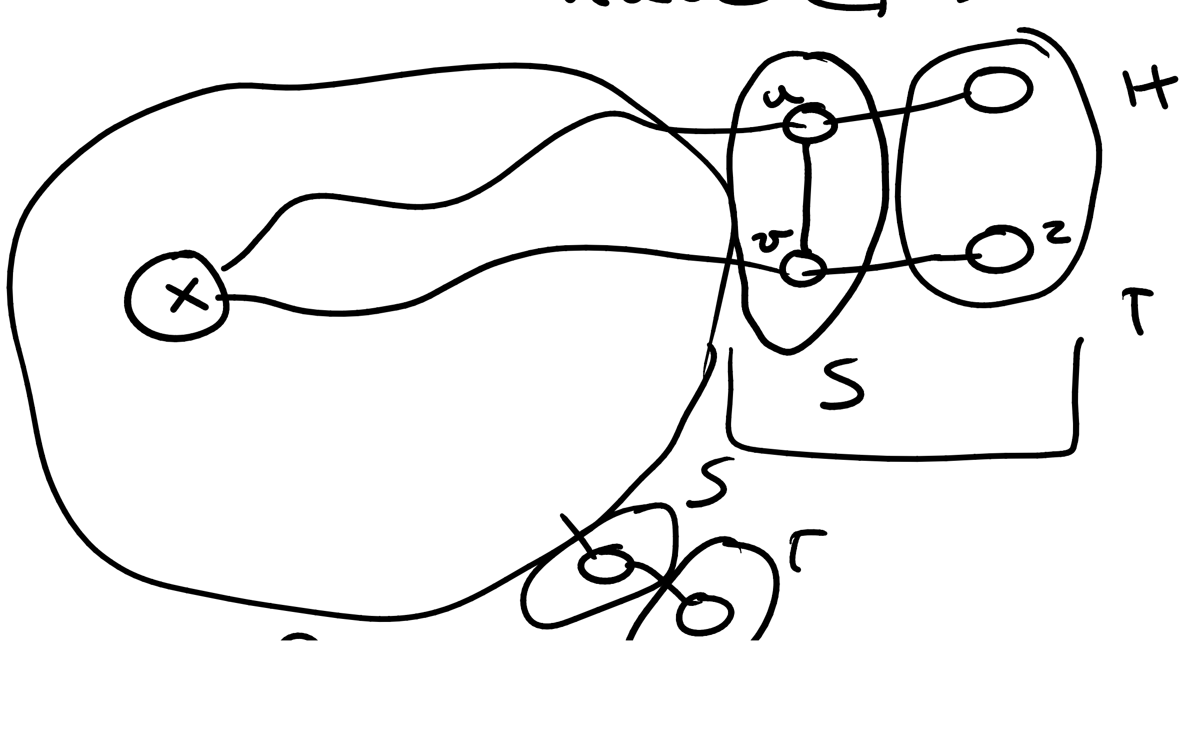
distance from x

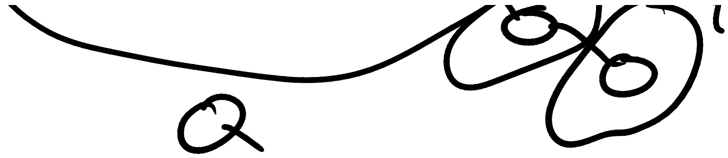
- define H = subgraph induced by $G[T]$
- define S = vertices in $G-T$ with neighbors in $V(H)$ or just T
- define Q = component of $G-S$ with v

Note: S must be a clique

→ has neighbors in Q and H $\forall w \in S$

→ any cycle from $H \to Q \to H$ passing through any $u, v \in S$ must have (u,v) as a chord





- define $G' = G[S \cup V(H)]$
- I.H. on G'
  - → consider some $u \in S$
  - → $\exists$ simplicial vertex in $H \to z$

↳ also simplicial in G $\square$

QEWL

Recall G is perfect if

$$\chi(G) = \omega(G)$$

Show: chordal graphs are perfect
- deleting vertices can't create chordless cycless

→ show $\chi(G) = \omega(G)$ for all

→ show $\chi(G) = \omega(G)$ for all chordal graphs

Note: chordal G has a SEO
- consider reverse order
- as v_i is added back to G_i, $N'(v_i)$ is a clique
- use greedy coloring on each v_i when we add them back
- If v_i gets color k, v_i belongs to clique K_k

→ etc.

→ we'll end up with some v_j with color l in clique of size l, where l is maximum over all of G

$$\Rightarrow \chi(G) = \omega(G)$$

Chordal graphs are perfectamundo □

QEWL