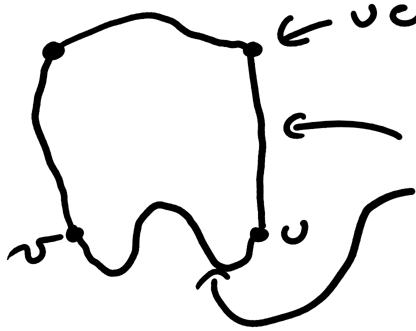
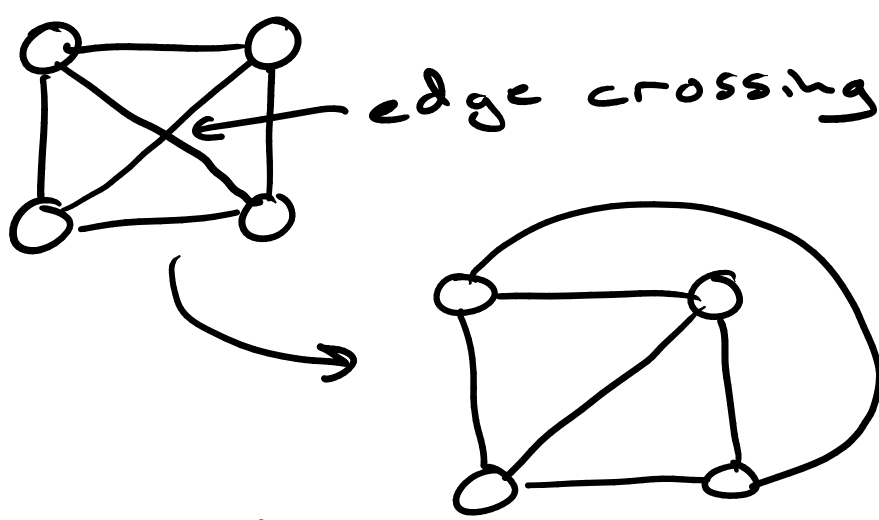
Graph drawing: a mapping of vertices in a graph to some point on the plane, and edges mapped to curves between vertices



← vertex as a point

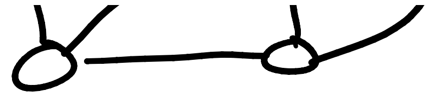
← each edge as a curve

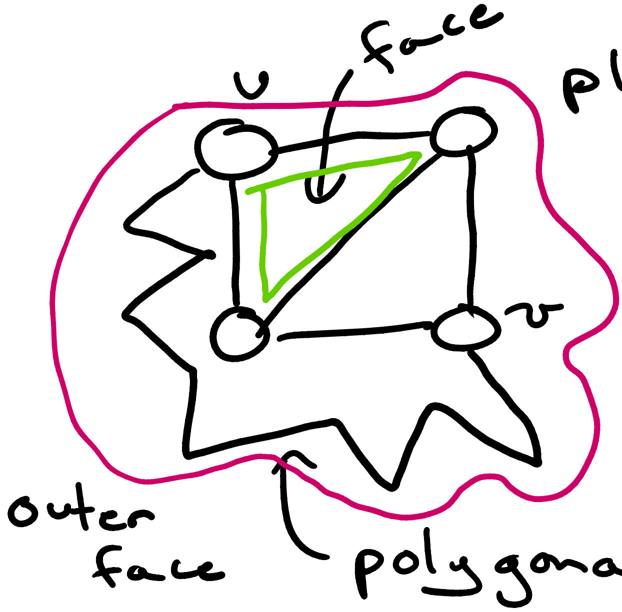
Graph planarity: a graph is planar if a drawing exists where no edges cross



← edge crossing

no crossings → planar





planar embedding
→ graph drawing w/o crossings

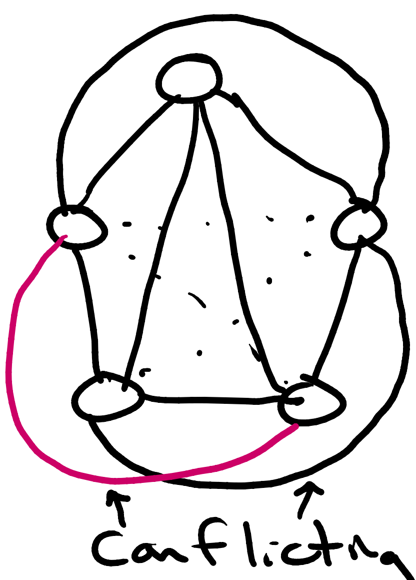
plane graph: graph with a planar embedding

Face: area enclosed by edges in same drawing

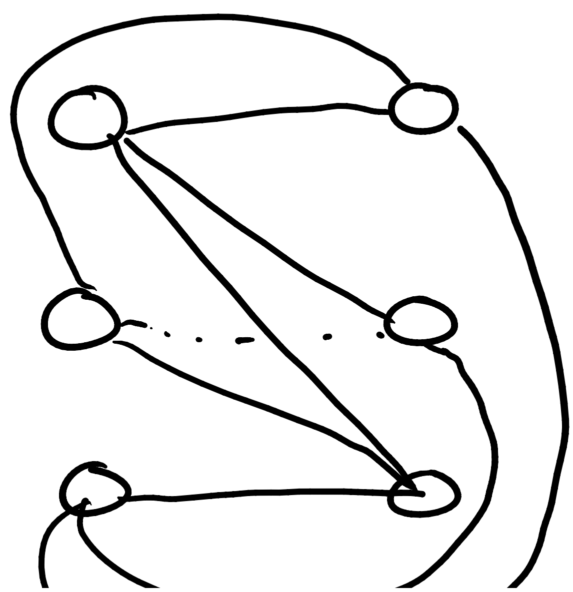
Outer face: the external/unbounded face of the drawing

Nonplanar graph examples

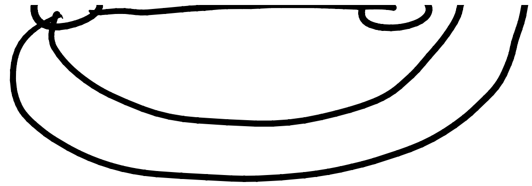
K_5



$K_{3,3}$

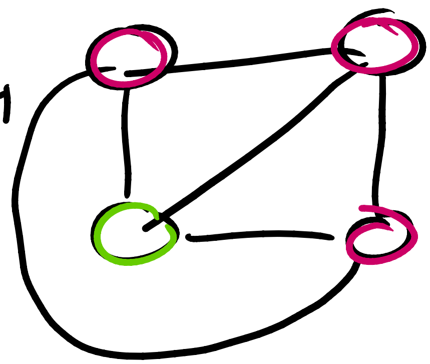


conflicting

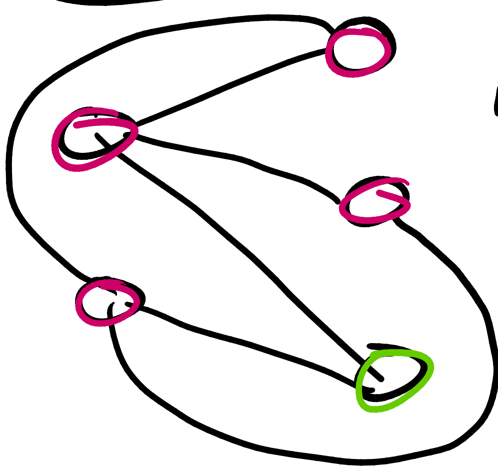


Outerplanar graph: planar graphs that have a drawing where all vertices are on the outer face

K_4



non-outerplanar



$K_{2,3}$

non-outerplanar

More examples: what is planar/outerplanar

Cliques?

K_5 is not

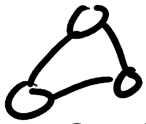
K is planar

cycles?

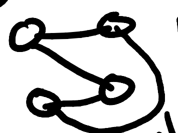
all planar and outerplanar

K_5 is not

K_4 is planar
not outerplanar

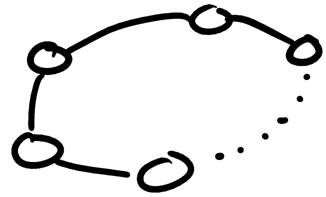


K_3 is both

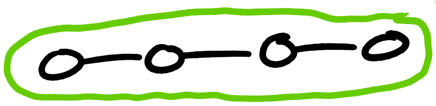


$K_{2,2}$ is both

all planar
and outerplanar

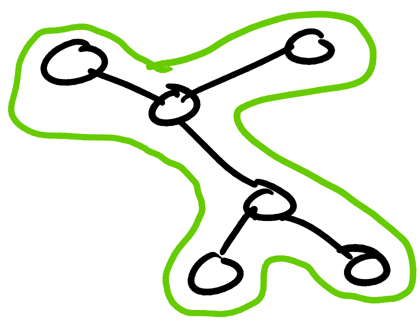


paths?



paths are both

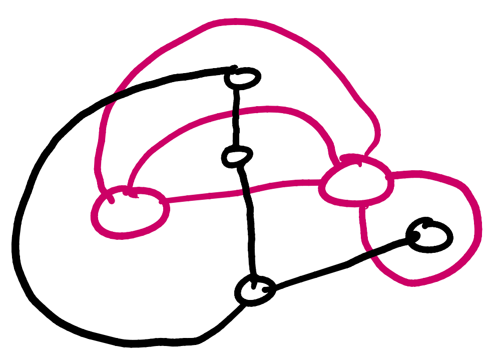
trees?



trees are both

Dual graphs

Dual graph G^* of plane graph G whose vertices are the faces of G and whose edges are defined based on the faces of G that share an edge



$(G^*)^*$

$$G \cong (G^*)^*$$

the dual graph of a dual graph is not always going to be isomorphic to the original graph

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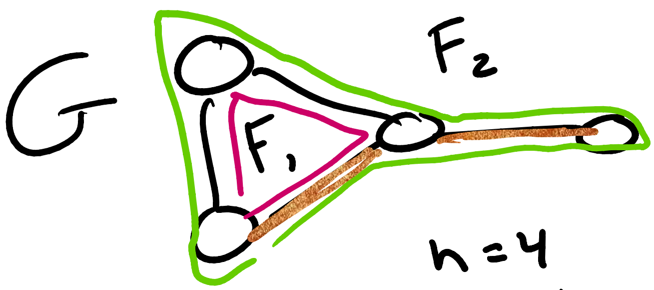
Faces of plane graphs



$n=4$
$e=4$
$f=2$

$\ell(F_1)=3$
$\ell(F_2)=5$

G has two faces

Length of a face is number of edges comprising the face (closed walk length of entire face)

Note: each edge contributes +2 to the total sum of all face lengths all face lengths

$$\sum_i \ell(F_i) = 2|E(G)|$$

G is bipartite $\Leftrightarrow$ all faces of G are even

$\Leftrightarrow$ G^* is Eulerian

G is bipartite $\Rightarrow$ all faces of G are even

Note: all possible closed walks on G are even

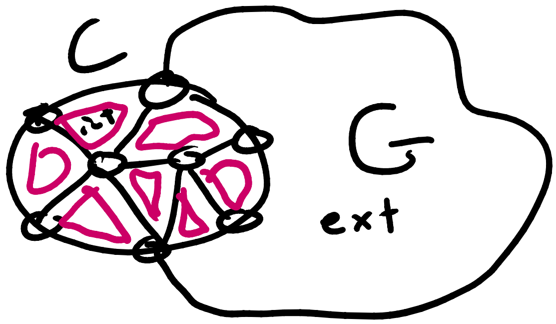
→ face lengths are defined by closed walk lengths

→ all face lengths are even ✓

All faces are even $\Rightarrow$ G is bipartite

- Consider some cycle $C \in G$ and an embedding of G
- The rest of G is either

- The rest of G is either internal or external to C



- Consider internal portion of G

→ all faces are even → $\sum \ell(F_i)$ will be even

Note: each internal edge counted twice in the sum

Note x2: each edge in C will be counted exactly once

PARITY => the cycle C must be even

→ holds for any choice of C ☑

All faces even <=> G^* is Eulerian

Note: vertex degrees in G^* are the length of each face the length of each face

→ all even

→ Eulerian $\square$

Euler's formula

$n - e + f = 2$ for connected planar graph G

n ← $|V(G)|$, e ← $|E(G)|$, f ← # faces

Let's prove via the POWER of induction on n

Basis $P(1)$ →

$n = 1$

$f = 2$ → holds

$e = 1$

Note: $f = e + 1$

Consider our $P(n)$ case

- Note: there exists some edge that is not a self loop
- contract that edge to get our $P(k)$ case

get our $P(k)$ case

I.H. on $P(k)$ case

$\rightarrow n' - e' + f' = 2$

$n' = n - 1 \rightarrow n = n' + 1$

$e' = e - 1 \qquad e = e' + 1$

$f' = f \qquad f = f'$

plug n' chug to check ✓

$(n - \cancel{1}) - (e - \cancel{1}) + f = 2$

$n - e + f = 2$ ✓ nate

If G is a simple connected planar graph with $|V(G)| \geq 3$, then $e \leq 3n - 6$

G is simple $\rightarrow \ell(F_i) \geq 3 : \forall F_i$

$2e = \sum \ell(F_i) \geq 3f$

$\underline{2e \geq 3f}$

consider $n-e+f=2$

$\rightarrow e = n+f-2$

$3e = 3n+3f-6$

$3e \leq 3n+2e-6$

$\Rightarrow e \leq 3n-6$

What if G is triangle-free?

$\ell(F_i) \geq 4$

$\sum \ell(F_i) \geq 4f$

$\Rightarrow e \leq 2n-4$

Note: if these inequalities don't hold $\rightarrow$ G is not planar

* Necessary, but not sufficient

A maximal planar simple graph G: adding

A maximal planar simple graph G: adding any edge to G results in a nonplanar graph

A minimal non-planar graph G: removing any edge from G results in a planar graph

Triangulation: a graph G whose planar embedding has all faces as triangles

G is max. planar graph $\Leftrightarrow$ $e = 3n-6$ and G is a triangulation

From before $\rightarrow$ $2e \geq 3f$

$\rightarrow \underline{e = 3n-6}$

we get equality when $2e = 3f$

$\rightarrow$ only happens when G is a triangulation

Note: a face length can only be increased when we can add an edge to obtain a larger planar graph

$e \leq 3n-6 \to e = 3n-6$

we hit equality $\square$