Lecture 22 - Kuratowski's Theorem

Friday, April 9, 2021 2:23 PM

Necessory conditions for planarity for G

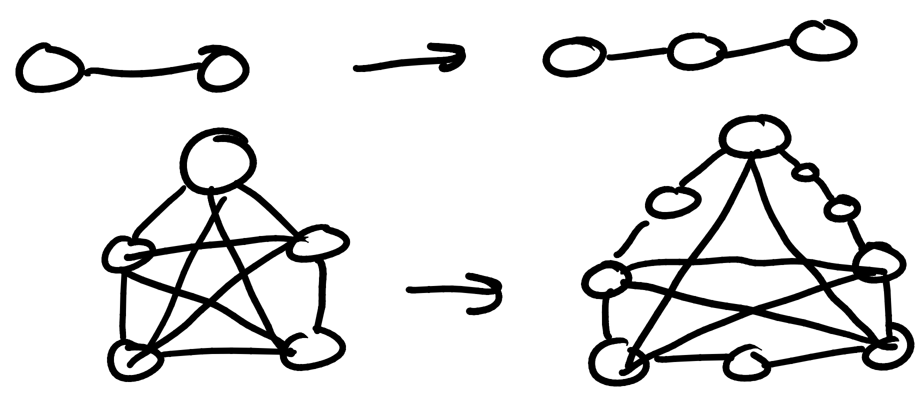
$m \leq 3n-6 \quad m = |E(G)|, \; n = |V(G)|$

$m \leq 2n-4$ if G is triangle-free

G has no $K_5, K_{3,3}$ subgraph

G has no $K_5, K_{3,3}$ subdivision

subdivided edge



$K_5, K_{3,3}$ subdivisions

$\longrightarrow$ Kuratowski subgraphs

Kuratowski's theorem

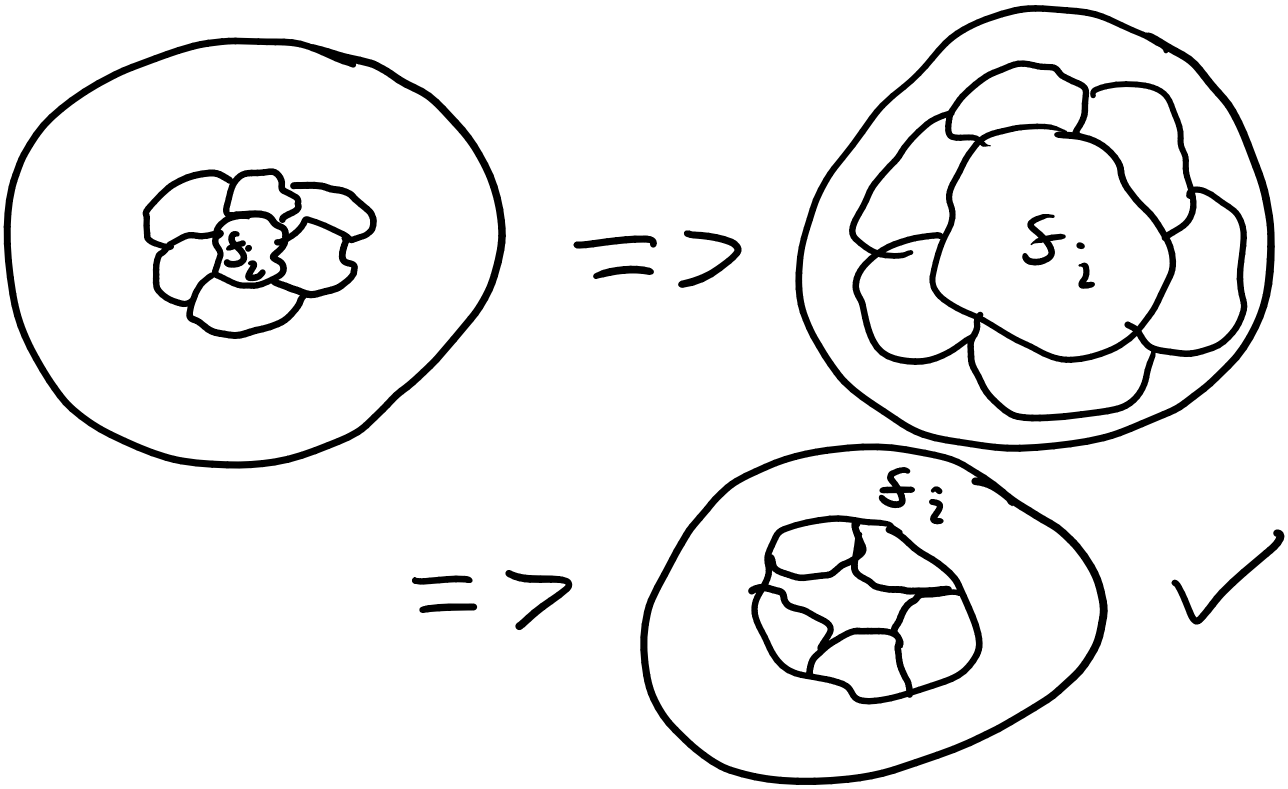
G is planar iff
G has no K. S.
(Kuratowski subgraph)

① For every face f_i of a planar embedding of G $\exists$ a planar embedding where f_i is on the outer face

- Consider some embedding of G on a sphere

Note: any embedding on a plane exists on some sphere

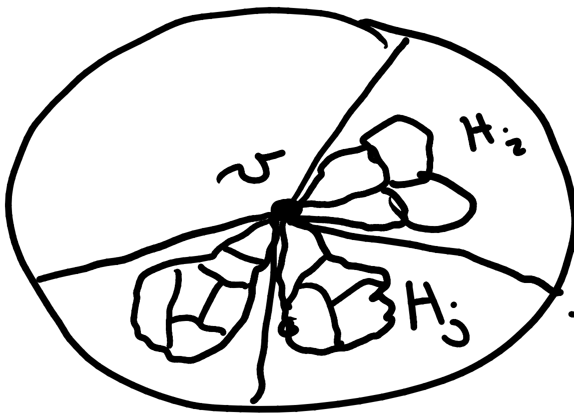
→ consider face f_i, and return projection of the return projection of the embedding bounded by f_i



(2) Every minimal nonplanar graph G is 2-connected

- $\forall H \subseteq G$, where $|H| < |G|$, H is planar
- Assume $\exists v \in V(G)$ s.t. $G - v$ is disconnected into $H_1, H_2 \dots H_k$ (all planar)

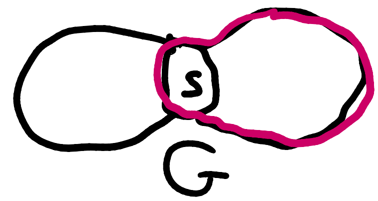
- Note: we can create a embedding of G by "squeezing" all H_i embeddings into $\frac{360}{k}$ degrees around v
- Note x2: from ①, $\exists$ for all H_i an embedding with v on the outer face



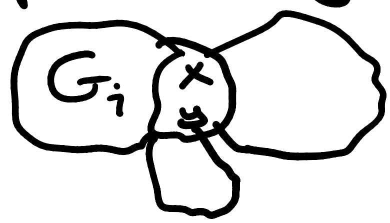
contradiction

→ G must be 2-connected

③ S-lobe: an induced subgraph of vertex set S and some component of G−S



Let $S = \{x, y\}$ be a separating set of 2-connected G. If G is of 2-connected G. If G is nonplanar => adding edge (x, y) to some S-lobe of G yields a non-planar graph

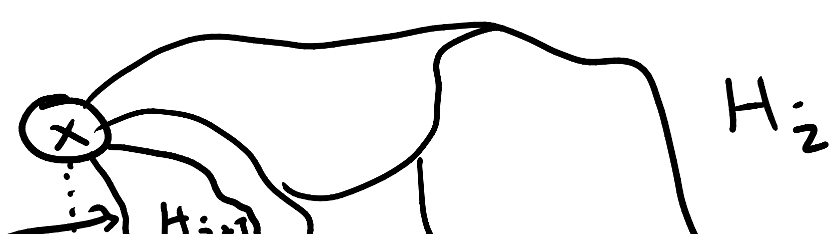


vertices edge

- Let $H_i = G_i \cup \{x, y\} \cup (x, y)$
- If H_i is planar, from 1 it has an embedding where (x, y) is on the outer face

- Assume that all H_i are planar

→ we can iteratively embed all $H_{i=2 \ldots k}$ into the face of H_{i-1}



Contradiction

$\Rightarrow$ at least one H_i is nonplanar ✓

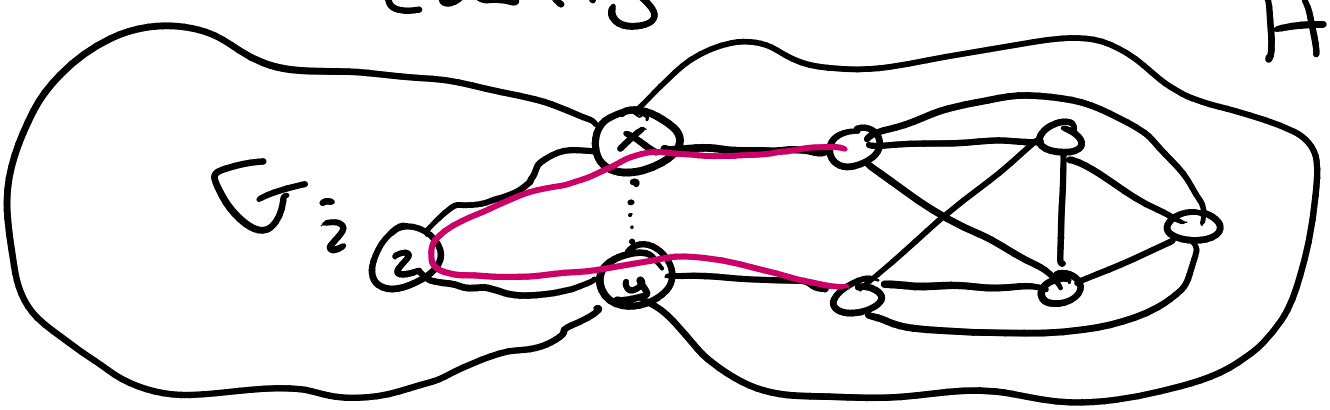
④ If G is a graph with the fewest edges among all nonplanar graph without K.S., then G is 3-connected

Note: G doesn't exist, but if it did, it would have to be 3-connected

→ restricting any possible counter-example to 3-connected graphs

– Note: deleting an edge can

- Note: deleting an edge can not create a K.S.
- $G-e$ is planar
- From (2), G is 2-connected
- Assume $\exists S = \{x, y\}$, then some S-lobe $G_i + S$ is nonplanar from (3)
- define H as that S-lobe $+ (x,y)$
- From our minimality condition, H must have a K.S.
- Consider our current configuration



However: Note that G_i is 2-connected

is 2-connected

$\exists z \in V(G_i)$, where z has paths to each of x, y

$\rightarrow$ we still have a K.S. when considering these paths

Contradiction

$\Rightarrow G$ must be 3-connected ✓

Next UP: show all 3-connected graphs w/o K.S. are planar

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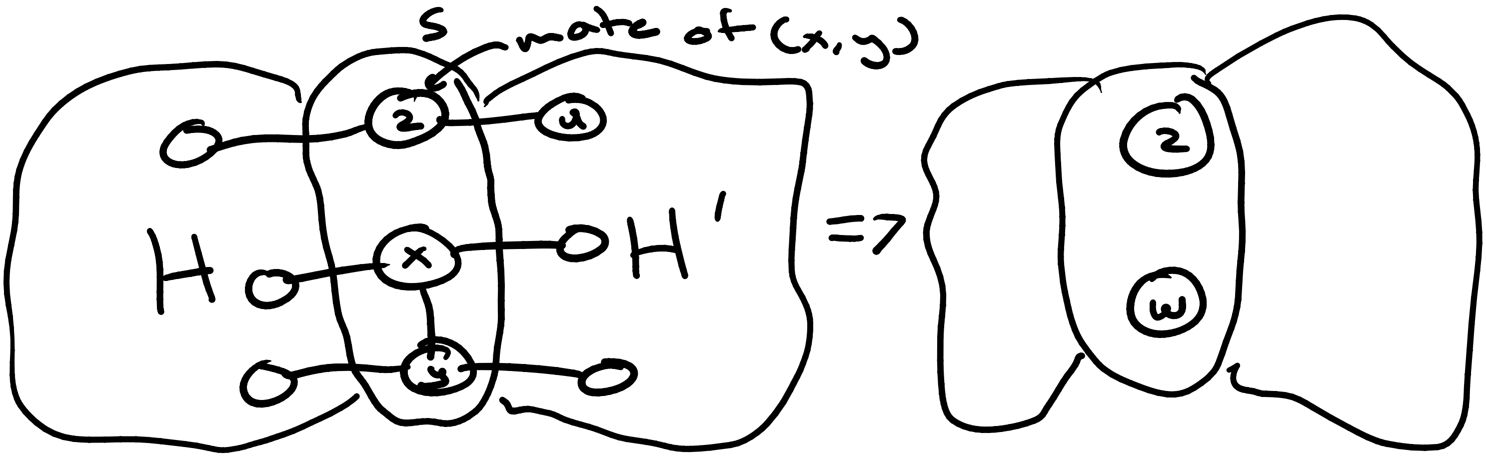
(5) If G is 3-connected and $|V(G)| \geq 5$, $\exists e \in E(G)$ s.t. $G \cdot e$ contracted is 3-connected

G • e contracted is 3-connected

Consider $e = (x, y) \in E(G)$ s.t. $G \cdot e$ is not 3-connected

$\rightarrow \exists\ S = \{x, y, z\}$



- Assume $\exists$ no such edge s.t. $G \cdot e$ is 3-connected
  $\rightarrow$ all edges are within a 3-separator with some 'mate' vertex

- Choose $S = \{x, y, z\}$ s.t. $|V(H)|$ is maximum

- Each of x, y, and z have a

- Each of x, y, and z have a neighbor in each of H and H'
- Consider $u \in N(z)$, $u \in V(H')$
- consider v, the mate of (u, z)
- Note: $G - \{u, v, z\}$ is disconnected
- $V(H) \cup \{x, y\}$ is connected and fully within same component of $G - \{u, v, z\}$

$\Rightarrow |V(H) \cup \{x, y\}| > |V(H)|$

x x x

Contradiction

x x x

$\Rightarrow \exists e \in E(G)$ s.t.

$G \cdot e$ is 3-connected ✓

(6) If G has no K.S. $\Rightarrow G \cdot e$ has

⑥ If G has no K.S. => $G \cdot e$ has no K.S.

contrapositive

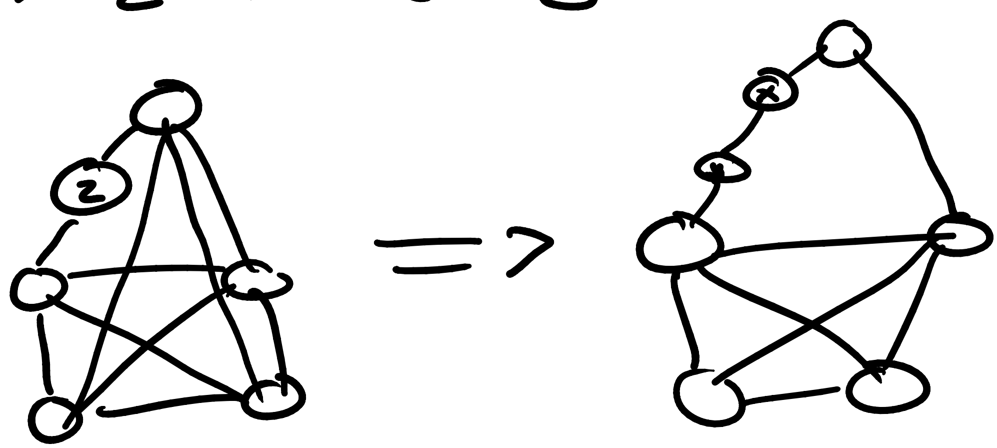
$G \cdot e$ has K.S. => G has K.S.

- define H as K.S. $\in G \cdot e$
- define $z \in V(G \cdot e)$, $z \leftarrow e = (x, y)$ contraction

Case 1: If $z \notin H$, then it obviously it holds

Case 2: $d(z) < 3$ (degree within H)

=> z is along a subdivided edge

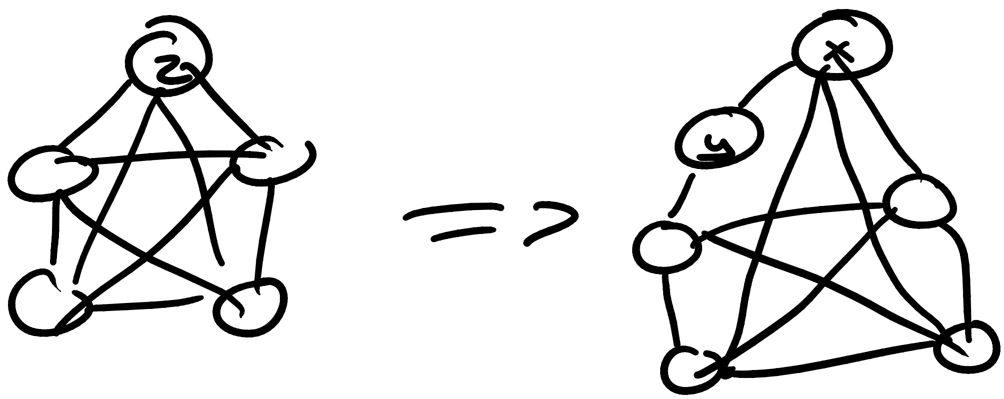


Case 3: $d(z) \geq 3$

$d(x) \leq 2$ or $d(y) \leq 2$

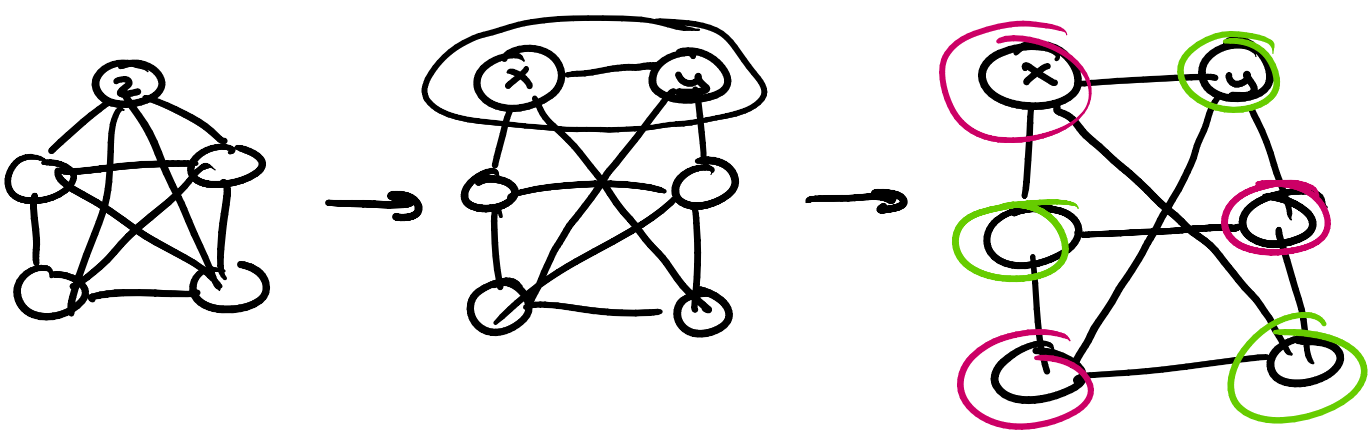
$d(x) \leq 2$ or $d(y) \leq 2$

$\Rightarrow$ same thing, z is along a subdivided path



Case 4: $d(z) \geq 3$, $d(x) \geq 3$ and $d(y) \geq 3$

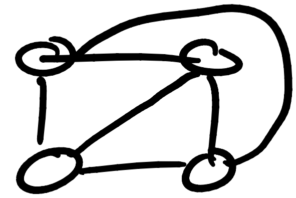
$\Rightarrow K_5 \to K_{3,3}$ is the only case



(7) If G is 3-connected with no K.S., G has a convex embedding on the plane

embedding on the plane

Induction on $|V(G)|$

Basis $P(4) \Rightarrow K_4$ 

Consider $P(n)$ case

- Note: $\exists e = (x, y)$ s.t. $G \cdot e$ is 3-connected (5)
- Note x2: if G has no K.S. then $G \cdot e$ has no K.S. (6)

Get $P(k<n)$ case via $G \cdot e$

- has no K.S. and is 3-connected

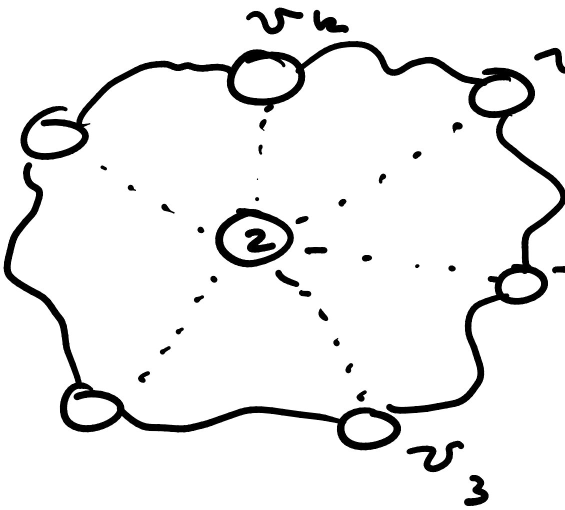
→ I. H. on $P(k)$ case

→ $P(k)$ has an embedding

- Consider $z \leftarrow (x, y)$
- Note: all $N(z)$ form a face that contains z

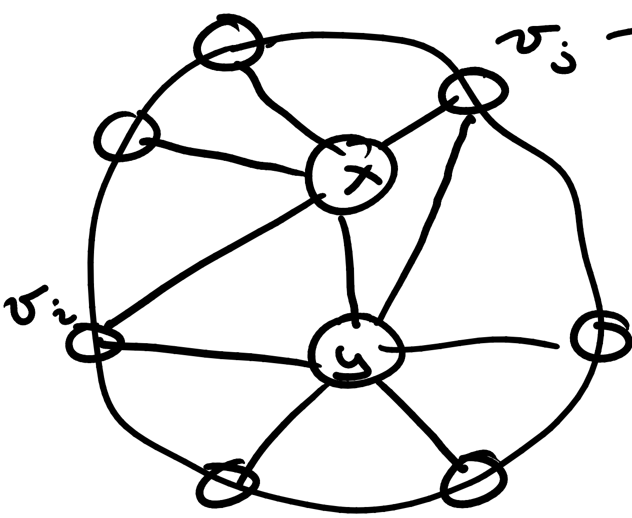
v_k

that contains z



- order all $N(z)$ as $v_1, v_2 \dots v_k$
- consider the following cases when we expand z back to x, y

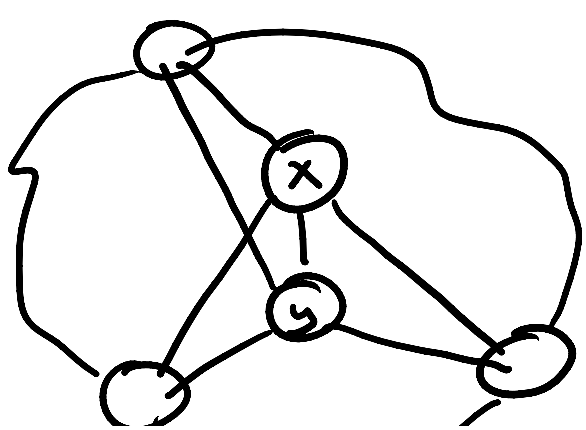
Case 1:



- $N(x)$ is some exclusive subset of $v_i \dots v_j$ and $|N(x) \cap N(y)| \le 2$

→ trivial to create a embedding for $P(n)$

Case 2: $|N(x) \cap N(y)| \geq 3$

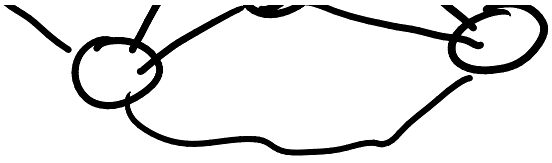


- we have K_5 K.S.

× ×

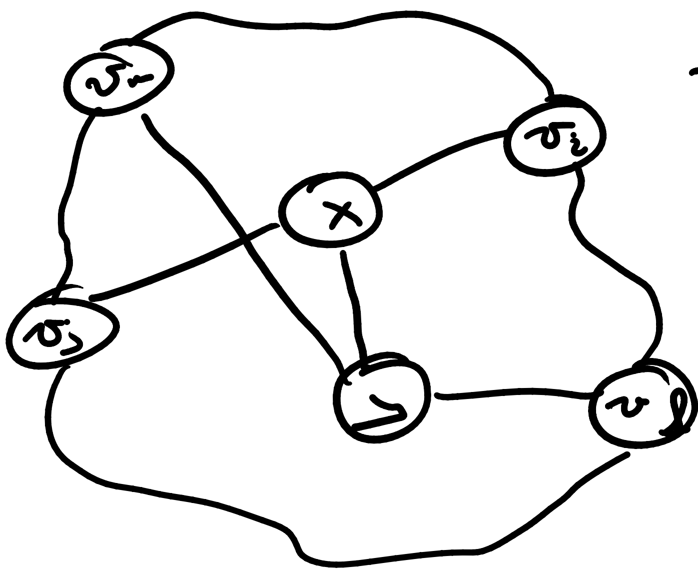
contradiction

× × ×



Case 3: $N(x)$ alternates with $N(y)$ s.t. $v_i v_j \in N(x)$
$v_\ell v_m \in N(y)$

$$v_i < v_\ell < v_j < v_m$$



- We have a $K_{3,3}$ K.S.

Contradiction

④ + ⑦ = Kuratowski's theorem

counter-example must be 3-connected

3-connected

no counter-example exists for 3-connected graphs

QEWL