Lecture 23 - Coloring of Planar Graphs

Tuesday, April 13, 2021 2:25 PM

Coloring of planar graphs

Map coloring

=> vertex coloring of a planar graph

How many colors?

→ What's the max chromatic number of planar graphs?

Can we bound the chromatic number as $\chi(G) \leq 5$?

5-color theorem: yes we can

Approach: find a minimum counter-example

Note: such a counter-example must have some v: $d(v) \geq 5$

→ $\exists v \in G$, G is min. counter-example where $d(v) = 5$?

Consider: $m \leq 3n - 6$

O.S.F.: $\sum d(v) = 2m$

If $\forall v \in V(G): d(v) = 6$

$2m = 6n$

$m \leq 3n - 6$

$2m \leq 6n - 12$

$6n \leq 6n - 12 \rightarrow$ NO

$\Rightarrow$ we have at least one vertex with degree < 6

5-color theorem: all planar graphs can be colored in 5 colors

Induction on $|V(G)|$

Basis $P(\leq 5) \rightarrow$ any graph with $|V(G)| \leq 5$ can be colored with 5 colors

$P(1)\ P(2) \ldots$

$P(n)$: Consider planar graph G

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- $\exists v \in V(G): d(v) \leq 5$

$P(k)$: We take $H = G - v$

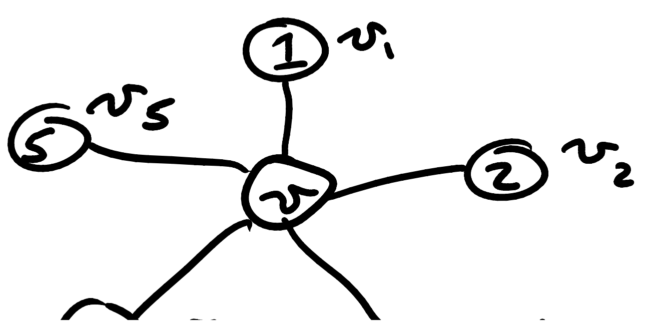
- I.H. on $H \rightarrow H$ has 5-coloring
- adding back v

Case 1: $d(v) \leq 4$

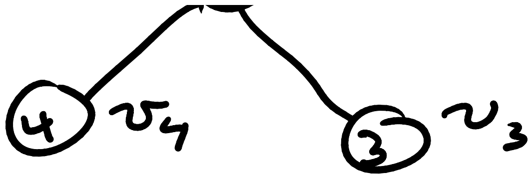
$\rightarrow$ can just color with 5th color

Case 2: $d(v) = 5$
and not all $N(v)$ have diff. colors $\rightarrow$ can color w/ 5th color

Case 3: $d(v) = 5$ all $N(v)$ have different colors

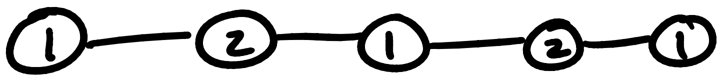


Show: this can be colored in 5-colors

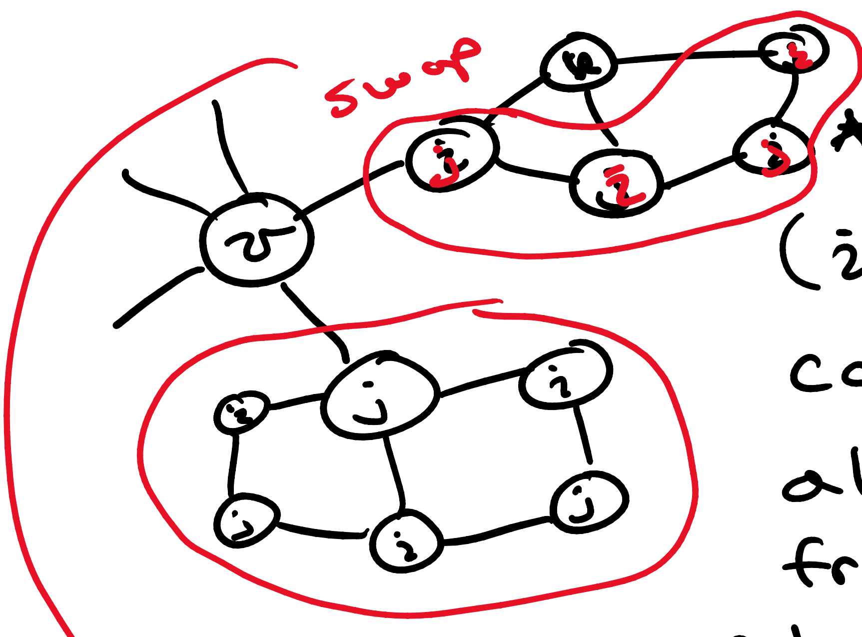


3-colors (i.e. 3 reducible)

- Kempe chains: color-alternating paths



- Consider all possible i,j color-alternating paths around v



→ we reduce $N(v)$ to 4 colors

★ If for a given (i,j) pair of colors, any i,j-alternating paths from $N(v)$ don't intersect → we can swap colors on one induced subgraph for one i,j

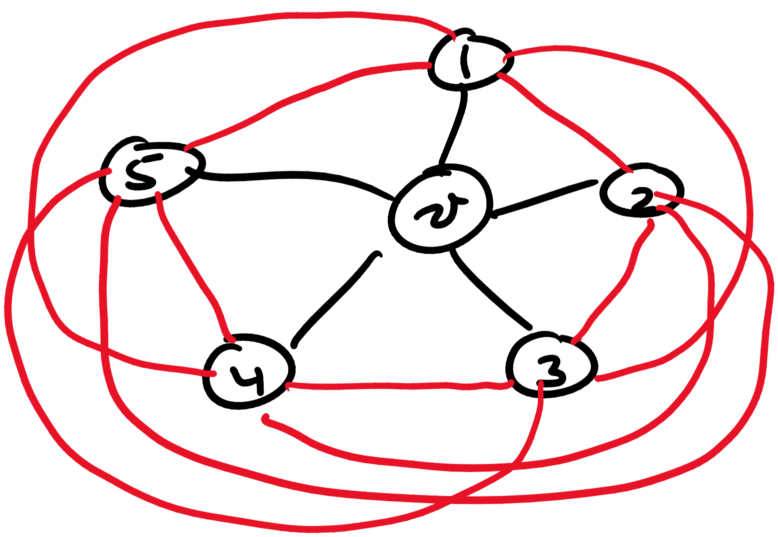
→ we can color v with the

5 th color

Else: for all possible (i, j) pairs

$\exists$ i, j-alternating path

from v_i to v_j



Note: as these paths exist $\to$ we have a K_5 Kuratoski subgraph

$\to$ $\exists$ at least one (i, j) pair

without that $v_i v_j$-color-alterating path $\square$ QEWL

What aboot 4-colors?

Can we do the same approach?

As we saw before: we're looking to find some minimum unavoidable configuration that a counter-example must have

→ if we can show all possible such configurations are reducible to color v with some 4th color, we can prove the 4-color theorem

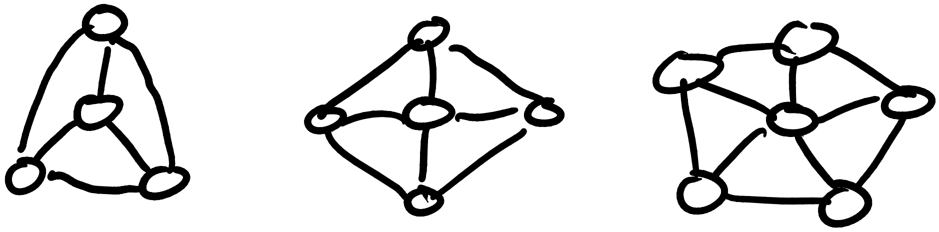
=> all planar graphs are 4-colorable

To help minimize possible configurations: consider triangulations

→ all planar graphs are subgraphs of a triangulation

If we can show all triangulations are 4-colorable → all planar are 4-colorable $\rightarrow$ all planar graphs are 4-colorable

From our 5-color proof:



our possible configurations

Note $\delta(G) \geq 3$ if G is a triangulation and $|V(G)| \geq 4$

Let's try our inductive proof for the 4-color theorem

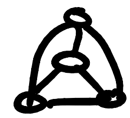
Basis $P(\leq 4) \rightarrow$ can trivially color

$P(n)$: We have G with some vertex $v : d(v) \leq 5$

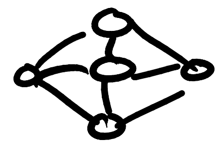
$P(k<n)$: Consider $H = G - v$

via I.H. $\rightarrow$ H is 4-colorable via I.H. → H is 4-colorable

- add back that v

Case 1: $d(v) = 3$ 

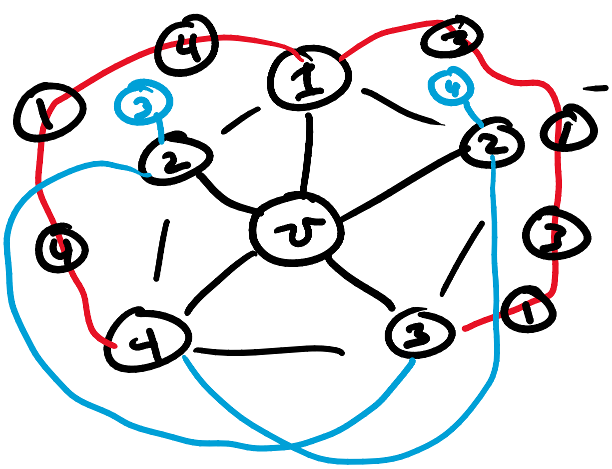
→ can just color v with 4th color

Case 2: $d(v) = 4$ 

→ yes we can via the same argument as with 5-color proof

Case 3: $d(v) = 5$

Note: exactly two vertices have the same color



- Consider paths from $1 \to 3$ and $1 \to 4$

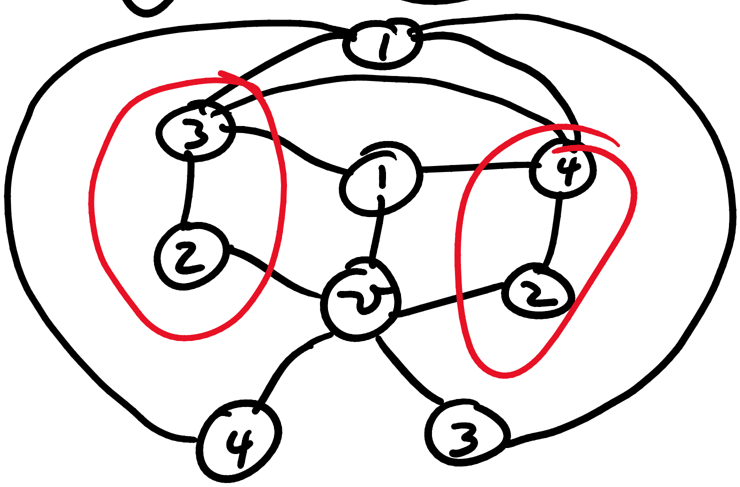
Note: we could eliminat color 1 if these paths did not exist

→ Now consider the vertices

colored with 2

We can't have a $2 \to 3$ alternating path $\to$ can swap colors on both of of the induced subgraphs containing color 2 to get only 3 colors in v's neighborhood $\square$

NOPE



From this: we need to consider larger possible configurations

How many? A LOT

Originally: 1800

Reduced: 633

4-color theorem actual proof:

computationally determine all possible minimum configuration and their reductions