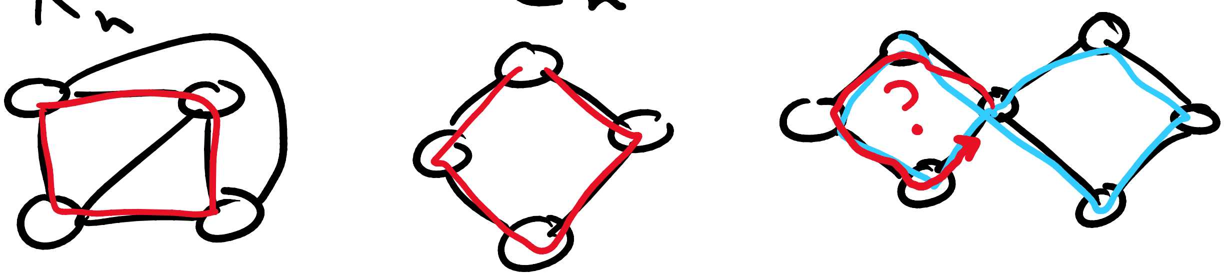
Hamiltonian graph: a graph containing a Hamiltonian cycle

Hamiltonian cycle: spanning cycle

Hamiltonian path: spanning path

What graphs are Hamiltonian?

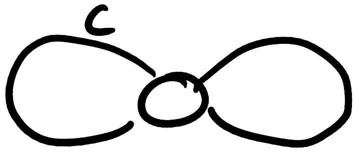
K_n C_n Eulerian



Necessary conditions:

- connected → trivial
- 2-connected → cycle can't pass through a cut vertex

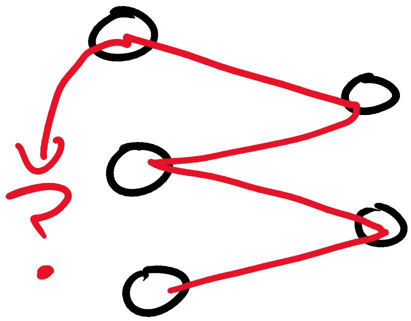
c



- If $G_{X,Y}$ is bipartite, $|X|=|Y|$

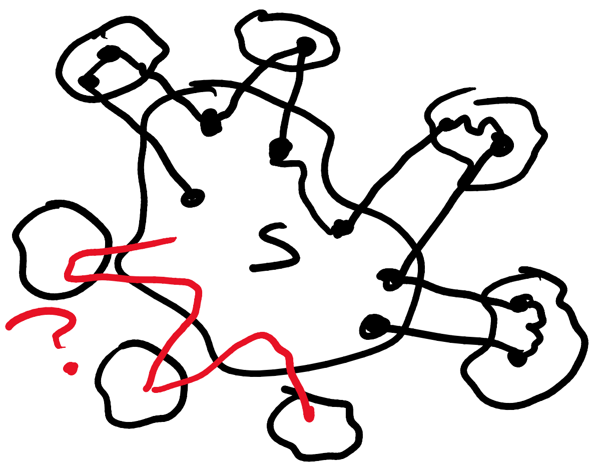
→ a cycle needs to have

→ a cycle needs to hop between sets an even number of times



- If $c(G)$ is # components of G

$$c(G-S) \le |S| \quad \forall S \subseteq V(G)$$



What about sufficient conditions?

if $|V(G)| \ge 3$ and $\delta(G) \ge \frac{|V(G)|}{2}$

- consider maximum non-Hamiltonian G'
  - → $G' + e$ = Hamiltonian graph
  - → G' has a spanning path

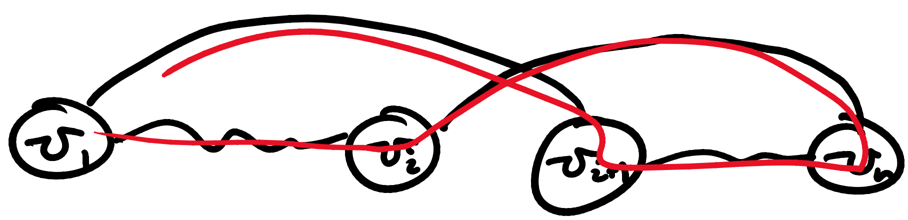
- consider this path in some order

$v_1 v_2 \dots v_n$

If along this path $\exists\, v_i v_{i+1}$

s.t. $v_i \in N(v_1)$, $v_{i+1} \in N(v_n)$

$\rightarrow$ we can create a cycle

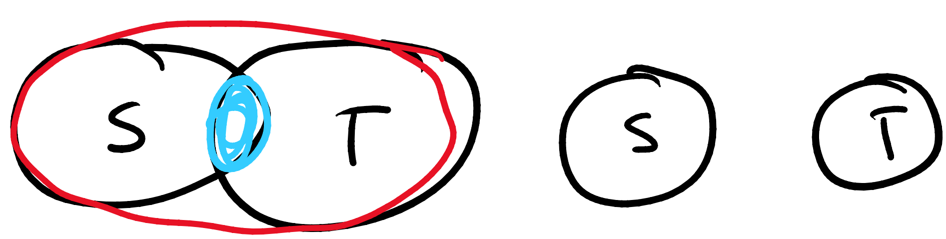


- define $S = \{ i : (v_1, v_{i+1}) \}$

$T = \{ i : (v_n, v_i) \}$

show: $|S \cap T| \geq 1 \rightarrow$ we have a cycle

$$|S \cup T| + |S \cap T| = |S| + |T|$$



$$|S| + |T| = d(v_1) + d(v_n) \geq |V(G)|$$

$|S \cup T| + |S \cap T| \geq |V(G)|$

$|S \cup T| < |V(G)|$ since no (v_1, v_n)

$\Rightarrow |S \cap T| \geq 1$

$\rightarrow$ Contradiction

$\Rightarrow$ we must have a spanning cycle $\square$

If $\forall u, v \in V(G)$ $(u,v) \notin E(G)$
$d(u) + d(v) \geq |V(G)|$

G is Hamiltonian iff
$G + (u,v)$ is Hamiltonian

$(\Rightarrow)$ trivial, adding an edge won't make $G + (u,v)$ non-Hamiltonian

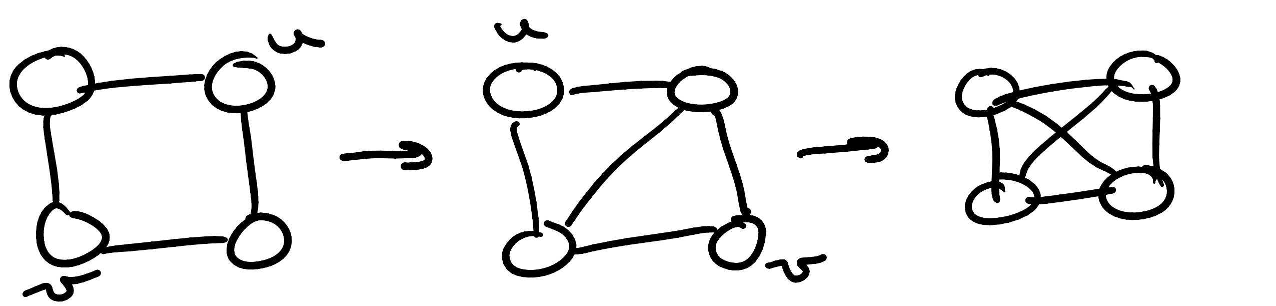
won't make $G + (u,v)$
non-Hamiltonian

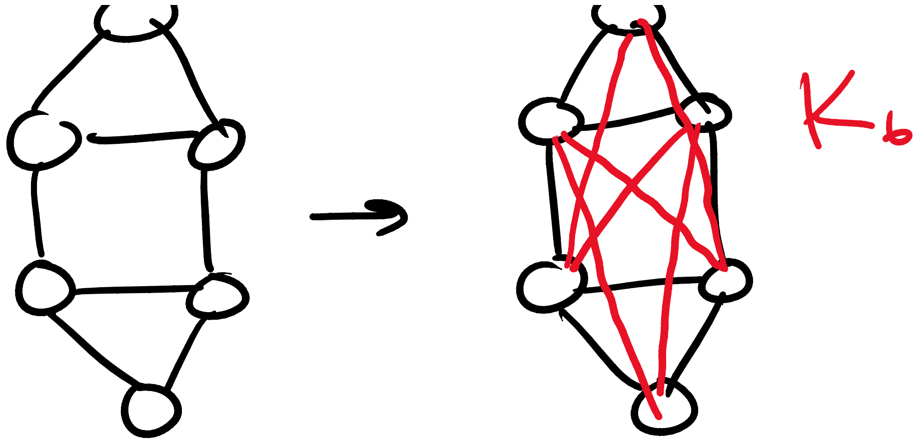
($\Leftarrow$) follows from our prior proof
$\exists$ spanning cycle w/o (u,v)
since $|N(u) \cap N(v)| \geq 1$

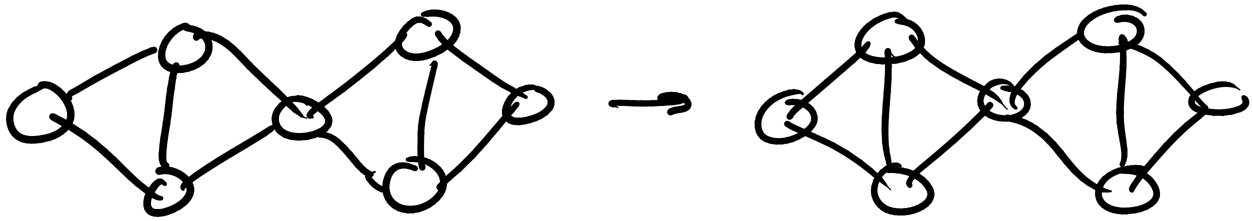
We can use the above to determine the <u>closure</u> of G

If G's closure is Hamiltonian
$\Rightarrow$ G is Hamiltonian

Closure of G:
add (u,v) $\forall u,v \in V(G)$
$d(u) + d(v) \geq |V(G)|$







→ G is Hamiltonian if its closure is Hamiltonian

→ Hamitonianness is preserved

Q: is the closure well-defined

→ does the order matter?

Consider:

- $e_1, e_2 \ldots e_i$ and $f_1 \ldots f_j$ are edges added to create $C(G)$ where $C(G) = G_e$, $C(G) = G_f$

where $C(G) = G_e$, $C(G) = G_f$

→ is $G_e \cong G_f$?

- since e_1 can be added for G_e it must also be eventually added for G_f as some f_k

- If any e_l depends on e_1, there is some f_m that depends on f_k that <u>will</u> be added

=> all the same edges will eventually be added to both G_e and G_f ☑

Can we define a numerical relation of degrees of G and the existence of G's closure being a known Hamiltonian graph like a clique?

Hamiltonian graph like a clique!

Yes we can

$\Rightarrow$ Chvátal's Condition

consider G with degrees

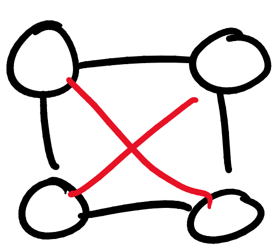
$$d_1 \le d_2 \le \dots \le d_n$$

if $i < \frac{n}{2}$ implies $d_i \ge i$

or $d_{n+1} \ge n - i$

$\Rightarrow C(G) \cong K_n$

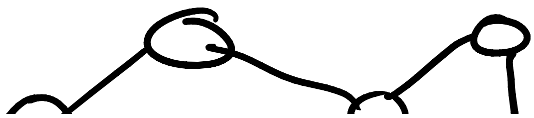
$\Rightarrow G$ is Hamiltonian



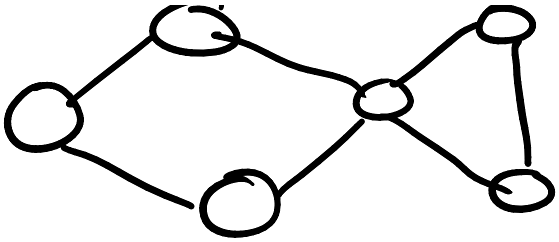
$d = 2, 2, 2, 2$ $\quad i = 1$

$i = 1, 2, 3, 4$ $\quad d_i > i$?

$2 > 1$? yes

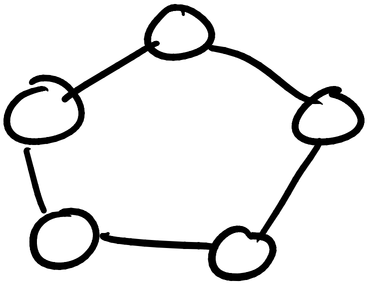


$d = 222224$ $\quad i:1$ $\quad 2 > 1$? ✓



$d = 222224$ $i:1$ $2>1$ ✓
$i = 123456$ $i=2$ $2>2$ ✗

$d_{n-i} \geq n-i$
$2 \geq 4$ ✗



$d = 22222$ $i=1$ $2>1$ ✓
$i = 12345$ $i=2$ $2>2$ ✗
$2 \geq 3$ ✗

→ Chvátal's condition is sufficient but not neccssary for Hamiltonianess

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Hamiltonian path → spanning path

<u>Graph join</u> between G and H, notationally $G \vee H$, is adding an edge between

is adding an edge between all $u \in V(H)$ to all $v \in V(G)$

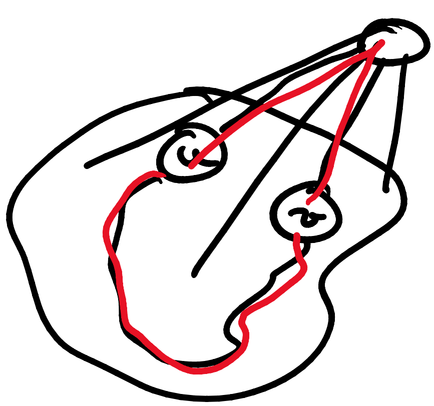
If $I = G \vee H$

$$V(I) = V(G) \cup V(H)$$

$$E(I) = E(G) \cup E(H) \cup \{\forall e = (u,v) \text{ s.t. } u \in V(H),\ v \in V(G)\}$$

$\Rightarrow$ G has a Hamiltonian path iff $G \vee K_1$ has a Hamiltonian cycle

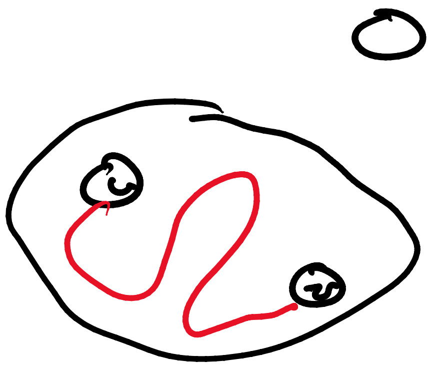
G has H.P. $\Rightarrow$ $G \vee K_1$ has H.C.



- consider a u,v- H.P. in G
- $G \vee K_1$ connects u and v through K_1 which creates

K_1, which creates a spanning cycle

$G \vee K_1$ has H.C. $\Rightarrow$ G has H.P.



Note: only 2 of K_1's edges are part of the H.C.

- wlg these edges attach to some $u, v \in V(G)$

$\rightarrow$ delete K_1 we'll have a H.P. from u to v ☑

QEWL STUFF

We can reconsider Chvátal's condition for Hamiltonian paths

G has degrees

$d_1 \le d_2 \le \dots \le d_n$

then

$i < \frac{n+1}{2}$ implies $d_i \ge i$

or $d_{n+1-i} \ge n-i$

$\Rightarrow$ G has Hamiltonian path