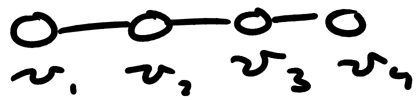
Errors:

Euler → Swiss
→ 1700s

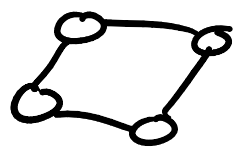
Königsberg → Russia

Types of graphs

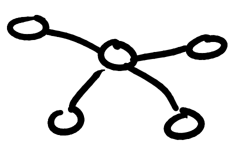
- Paths 

$\{v_1 v_2 v_3 v_4\}$

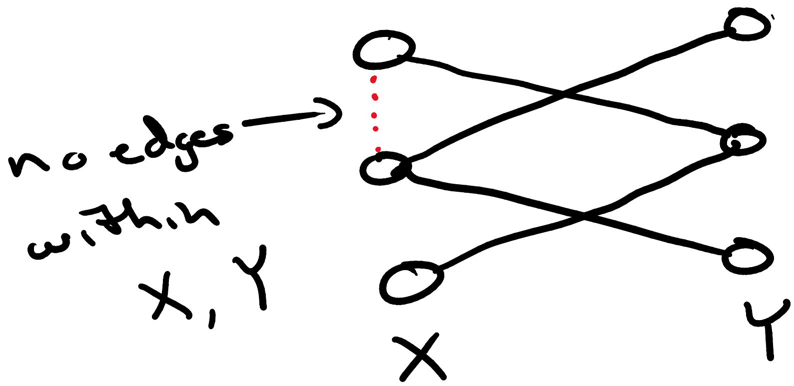
  P_n = path of n verts

- Cycles  $\{v_1 v_2 v_3 v_4 v_1\}$

  C_n = cyle of n verts/edges

  (connected)
- Trees  acyclic graphs

- Bipartite graphs: a graph containing at least two vertex disjoint sets, s.t. no edges exist between vertices in the sets the sets



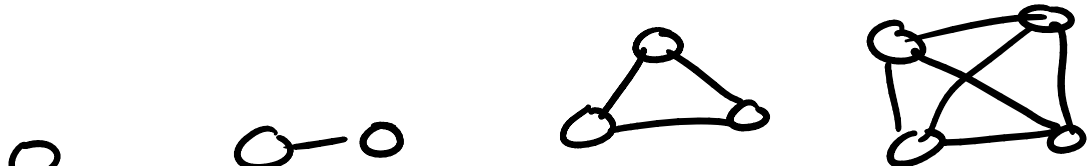
$B_{X,Y}$ = bipartite graph with sets X, Y

- Complete graph a.k.a. clique:

  a simple graph where all possible edges exist

  $$\forall u, v \in V(G) : \exists e = (u, v) \in E(G)$$
  $$u \neq v$$

  K_n = clique on n vertices



K_1 K_2 K_3 K_4

- Complete bipartite graph

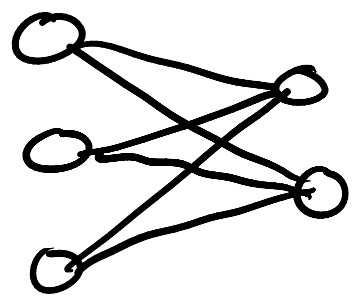
  $$\forall u \in X, \forall v \in Y : \exists e \in E(G)$$

  $K_{n\,m}$ = complet bipartite graph

$K_{n,m}$ = complete bipartite graph
where w.l.o.g. $|X| = n, |Y| = m$

without loss of generality

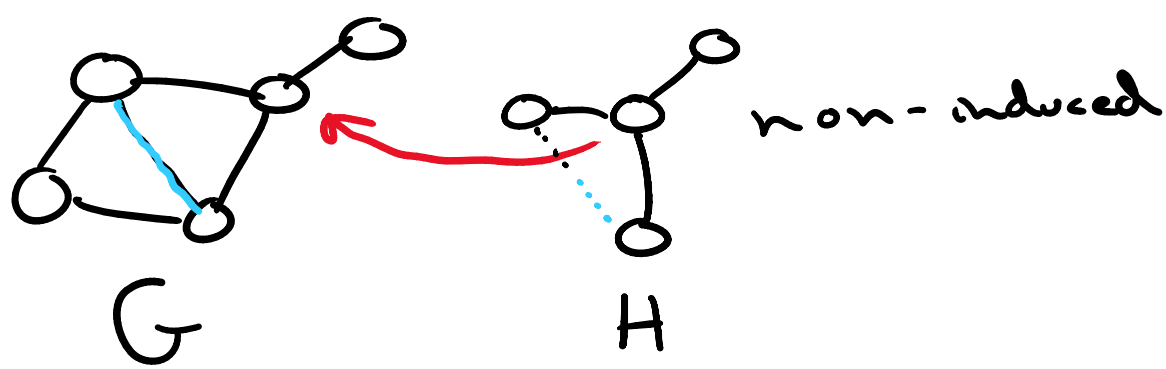
$K_{3,2}$



- subgraph: H is a subgraph of G if H is entirely contained in G

$\forall v \in V(H) \rightarrow v \in V(G)$

$\forall e \in E(H) \rightarrow e \in E(G)$

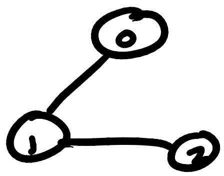


Adjacency matrix representation

A: $n \times n$ matrix that represents same graph G same graph G

if a_{ij} is nonzero in A $\Rightarrow$ edge$(i,j) \in E(G)$

row column

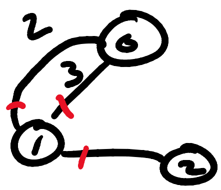


G

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \boxed{0} & \boxed{1} & \boxed{0} \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{array}$$

A

$\sum \text{row}_i = d(i)$

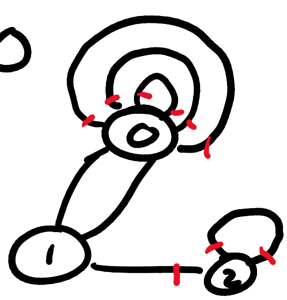


$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 5 & 0 \\ 1 & \boxed{5} & \boxed{0} & \boxed{1} \\ 2 & 0 & 1 & 0 \end{array}$$

A

$\underbrace{\sum \text{row}_1 = d(1) = 3}_{\text{not hold}}$

for weighted graphs



$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 6 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \end{array}$$

A

$\sum \text{row}_2 = d(2) = 3$

Graph Isomorphism

$G = (V, E)$ is isomorphic to $G' = (V', E')$

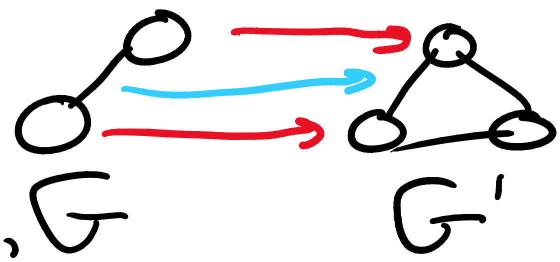
if there exists a one-to-one mapping

$\forall v \in V(G) \to u \in V(G')$

$\forall v \in V(G) \to u \in V(G')$

$\forall e \in E(G) \to f \in E(G')$

$\Rightarrow$ all vertices/edges in G are mapped to all vertices/edges in G'



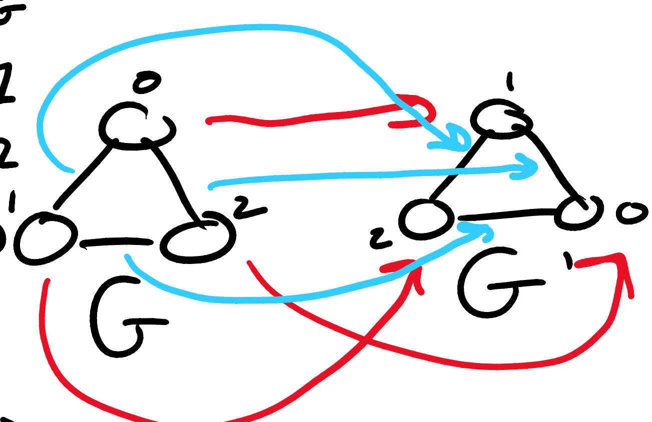
NOT isomorphic

$G \to G'$

$f(0) = 1$

$f(1) = 2$

$f(2) = 0$



YES isomorphic

$S(G) = \{2, 2, 2\}$
$= S(G')$

$f(0,1) = (f(0), f(1))$
$= (1,2)$

isomorphic

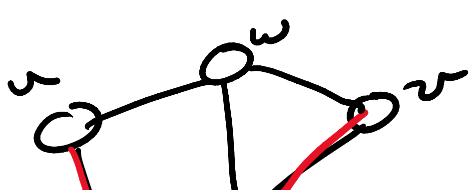
If $G \cong G'$

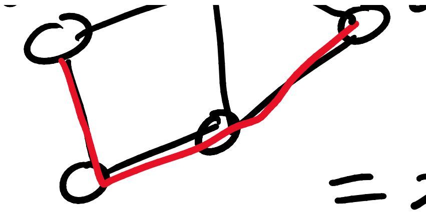
$\Rightarrow |V(G)| = |V(G')|$

$|E(G)| = |E(G')|$

$\Rightarrow$ sorted <u>degree sequences</u> are identical

list of degrees for all vertices





↳ list of degrees for all vertices

$\Rightarrow$ the <u>diameters</u> are identical

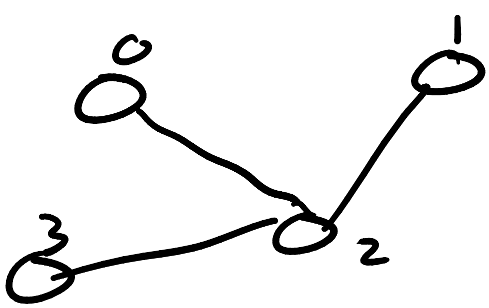
↳ longest shortest path

$\Rightarrow$ the <u>girths</u> are identical

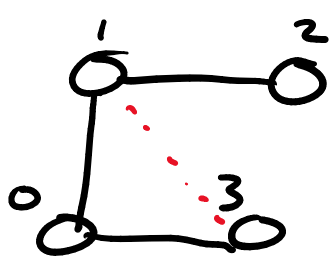
↳ shortest cycle

Note: these properties are <u>necessary</u> but not <u>sufficient</u>

$G \to G'$



G

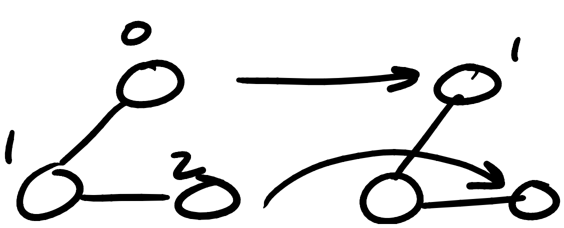


G'

~~$f(0) = 1$~~
~~$f(1) = 2$~~
~~$f(3) = 0$~~
~~$f(2) = 3$~~

$(0, 2) \to (f(0), f(2))$

$G \to (1, 3)$
G'



$f(0) = 1$
$f(1) = 0$

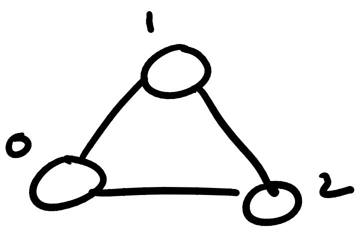
$f(1) = 0$

$f(2) = 2$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \quad \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{A'} \qquad PAP^T = A'$$

Automorphism

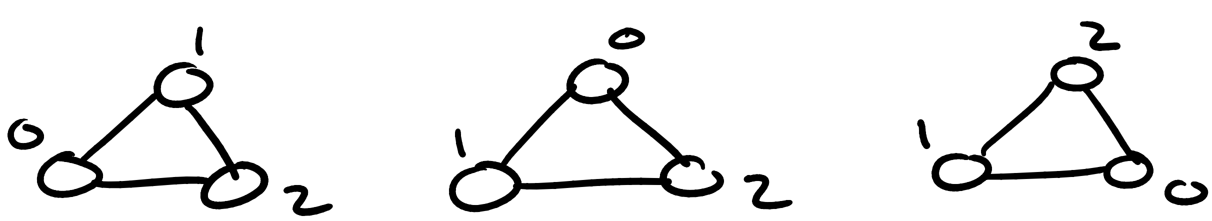
→ an isomorphic mapping of G to itself (s.t. the edge list is preserved)



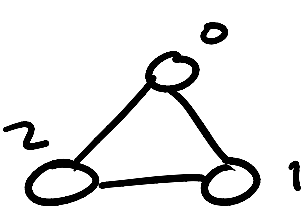
K_3

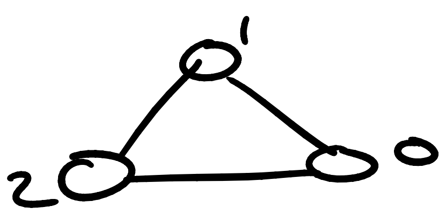
$E = \{(0,1), (0,2), (1,2)\}$

K_n: we can map any vertex to any other vertex



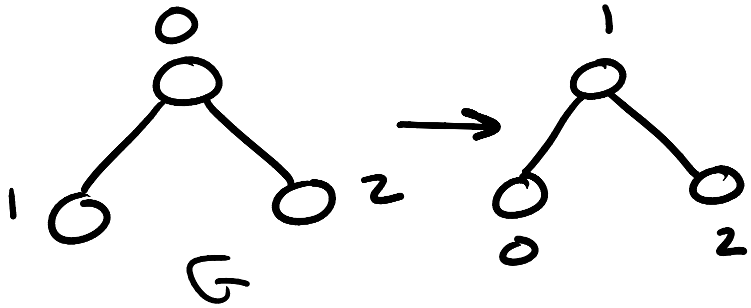
$f(u) \to v$





$\Rightarrow 6$ automorphisms of $K_3 \Rightarrow 3!$

$K_n \Rightarrow n!$



$f(0) \to 1$

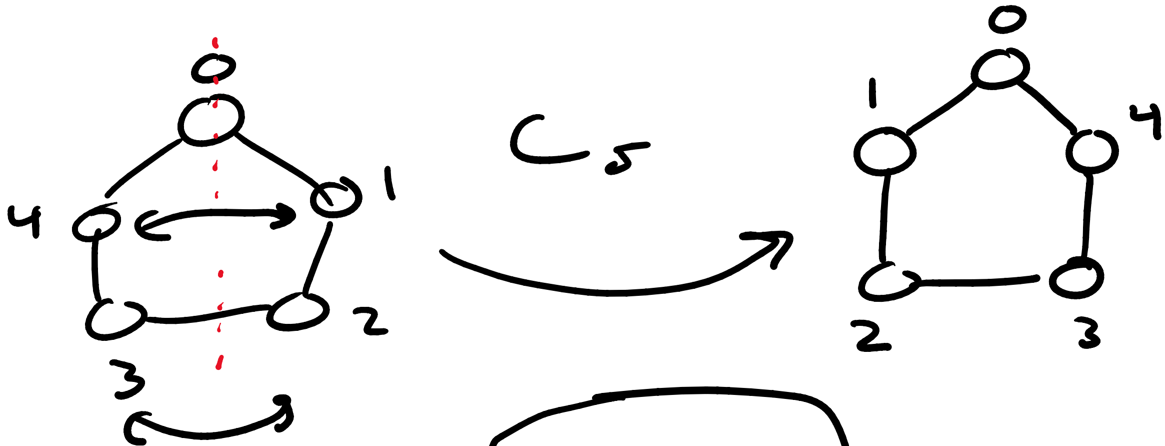
$f(1) \to 0$

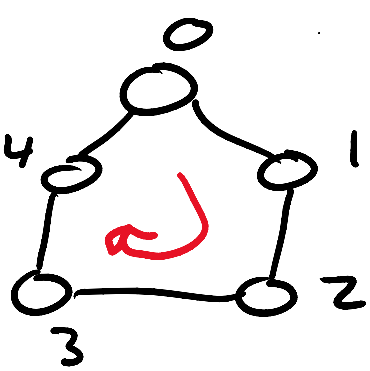
$f(2) \to 2$

NOT an automophic mapping

$E = \{(0,1), (0,2)\}$

$E' = \{(0,1), (1,2)\}$





$f(0) \to 4$

$f(1) \to 0$

$f(2) \to 1$

$f(3) \to 2$

$f(4) \to 3$