More definitions

complement of $G \rightarrow \overline{G}$

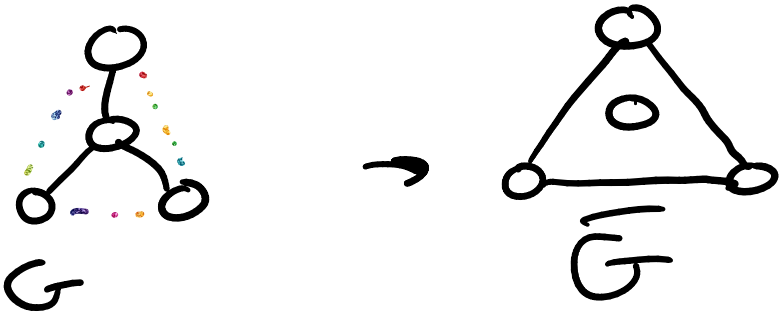
(G is simple)

$V(\overline{G}) = V(G)$

such that

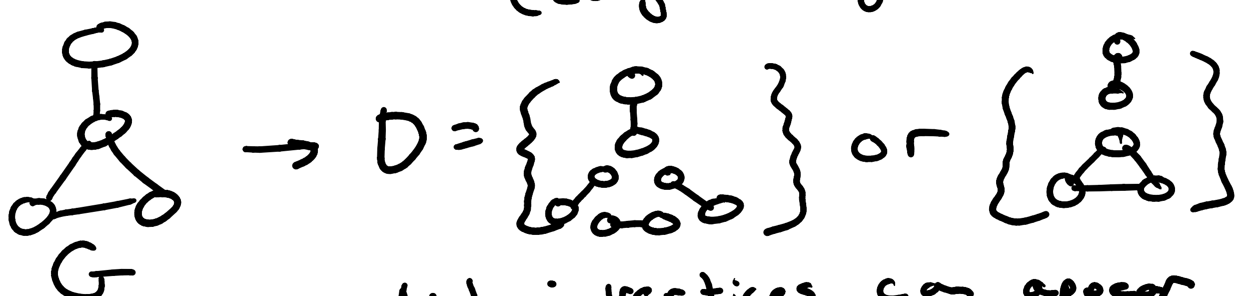
$E(\overline{G}) = \{\forall u, v \in V(G) : (u,v) \notin E(G)\}$

not in



decomposition of G

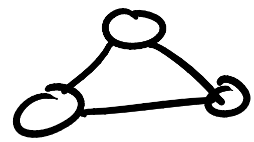
→ set of subgraphs s.t. each edge of G appears exactly once within this set (edge-disjoint)



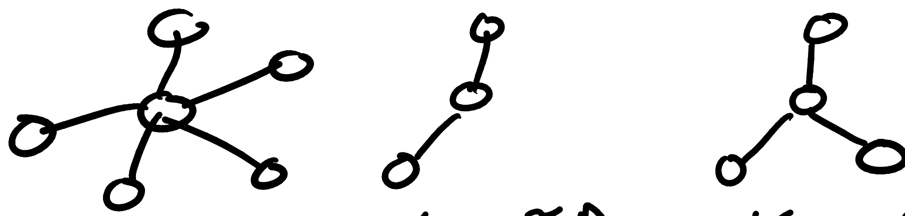
Note: vertices can appear a arbitrarily large number of times

a arbitrarily large number
of times

Triangle graph $\cong K_3 \cong C_3$



Star graphs: complete bicliques
with one bipartite set
of cardinality one



$K_{1,5}$ $\quad$ $K_{1,2} \cong P_3$ $\quad$ $K_{1,3} \cong$ **claw**

Time for a stroll (walk)

Walk: a list of vertices and/or
edges, s.t. each listing is
adjacent/incident to the
listings preceeding and
proceeding it

$W: \{v_0, e_5, v_3, e_6, v_5, e_{11}, v_0, e_5, v_3\}$

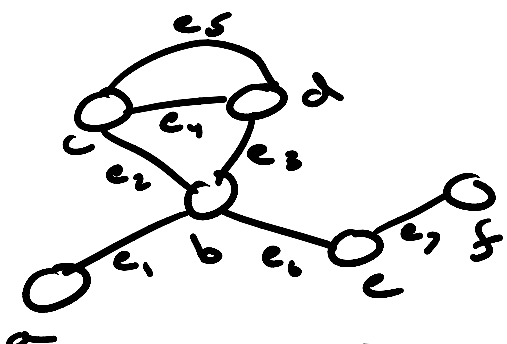
$W: \{v_0, e_5, v_3, e_6, v_5, e_{11}, v_0, e_5, v_3\}$

$W: \{e_5, e_6, e_{11}, e_5\}$

Note: we can repeat vertices and edge

Trail: as above, but we don't repeat edges

Path: as above, but we don't repeat edges or vertices



$W: \{a\, e_1\, b\, e_1\, a\, e_1\, b\, e_3\, d\, e_5\, c\, e_2\, b\, e_3\, d\, e_3\, b\, e_6\, e\}$ a,e-walk

$T: \{a\, e_1\, b\, e_3\, d\, e_5\, c\, e_2\, b\, e_6\, e\}$ a,e-trail

$P: \{a\, e_1\, b\, e_6\, e\}$ a,e-path

P_3 o–o–o length = 2

Length: number of traversed edges

Hop: traversal of a single edge

u,v-path (trail, walk): a path that starts at u and ends at v

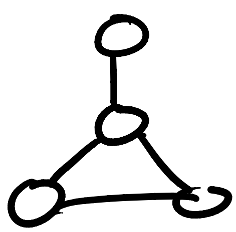
trail
walk
and ends at v

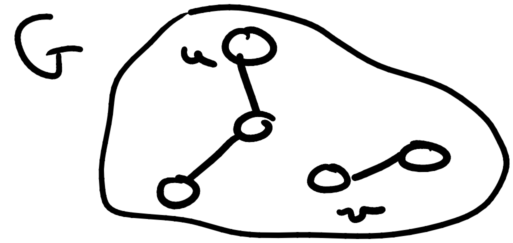
closed path: a path that starts and ends at the same vertex
trail
walk

Connectivity

aka Let's get connected
aka Let's connect the dots
aka vertices

G is connected if $\forall u, v \in V(G)$: $\exists$ u,v-path





G is connected

G is disconnected

connected components: a maximal connected subgraph of G

maximal: can't be made larger

maximum: the largest of possibilities

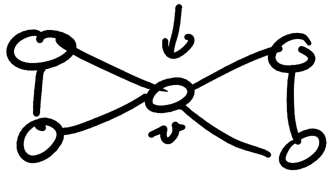
maximum: the largest of possibilities

Note: same for minimal / minimum

but smaller / smallest

cut-vertex: some $v \in V(G)$ s.t. $G - v$ has more connected components (CCs) than G

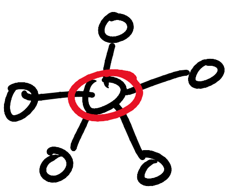
cut vertex



v

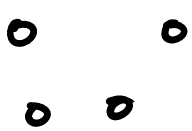
G

$G - v$



connected component

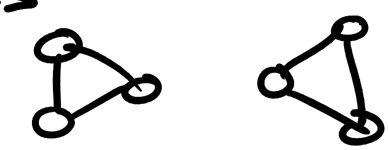
cut-edge: some $e \in E(G)$ s.t. $G - e$ has more CCs



cut edge

e

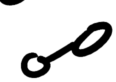
G

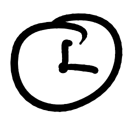


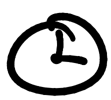
$G - e$

P_1

P_2

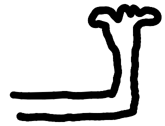


Time  for the meat

aka  4 

aka **induction**

aka induction

Weak induction 

Prove: $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$

Base: $P(n=1) \Rightarrow 2^1 = 2^{1+1} - 2 = 2$ ✓

Inductive Step: $P(n = k+1)$

Inductive hypothesis: $P(k)$ is true

show $P(k+1)$ is true

$$P(n=k+1) = \underbrace{2^1 + 2^2 + \ldots + 2^k}_{\text{this is } P(k) \text{ case}} + 2^{k+1}$$

if we assume $P(k)$ holds

$$P(n=k+1) = \underbrace{2^{k+1} - 2} + 2^{k+1}$$

$$P(n=k+1) = \underbrace{2^{k+2} - 2} \quad \square$$

Weak induction $\xrightarrow{\text{inductive step}}$

$P(1)$, $P(2)$, $P(3)$ ... $P(k)$, $P(k+1)$

show basis holds — assume holds via I. H. — show this holds

holds via I. H. this holds

inductive hypothesis

Strong induction  ← arm

P(1), ..., P(k), ..., P(n)

show basis — assume for all $1 \le k < n$ — prove this holds

Example proof:

length is odd

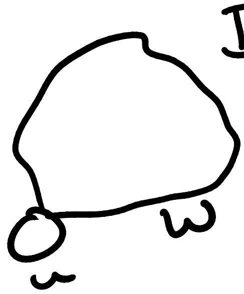
Show every closed odd walk contains an odd cycle

C_n: n = odd

Induction on l = length of odd walk

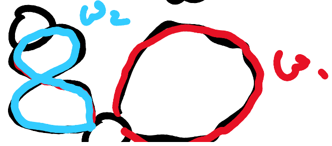
Basis $P(l=1)$: 

Inductive step: $P(n > k \geq 1)$



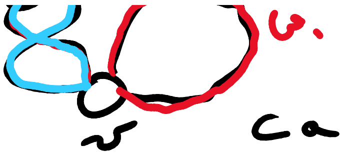
Assume we have a walk of length n, n = odd, walk is closed

Consider cases:



case 1: no vertices repeat on walk

trivially → walk is a cycle



trivially → walk is ... v

case 2: some vertex v repeats

↳ implies we can separate walk

$W \to W_1 + W_2$

consider lengths of W_1 and W_2

→ one of W_1 or W_2 must be odd

★ parity argument ★

wlog $|W_1|$ is odd

Note: $|W_1| < |W|$

Power of strong induction

→ $P(k) = W_1$

via I.H. → W_1 contains an odd cycle

Note: initial assumptions are valid for $P(k)$ case

So: if W_1 has C_{odd} then W has C_{odd} as $W_1 \in W$ ☑

Construct an inductive proof on the number of edges m = |E| of a graph with n = |V | vertices to prove the following: A connected simple undirected graph must have at least m = (n – 1) edges. You can use strong or weak induction for your proof.

induction on number of edges

Basis $P(1)$:  $m=1$ $\quad m \geq n-1$

$n=2$ $\quad 1 \geq 2-1$

Assume $P(k)$ holds via I.H. $\quad 1 \geq 1$ ✓

show $P(k+1)$ holds

→ to formulate argument:

how can we add an edge

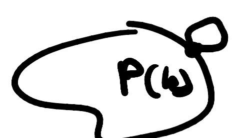
to some G w/ k edges?

Case 1: add two new vertices plus

that edge 

→ invalid, not connected

Case 2: add egdge as self loop



→ invalid, not simple

Case 3: we add an edge between

existing vertices

$m = k+1$ edges

via I.H. on $P(k)$

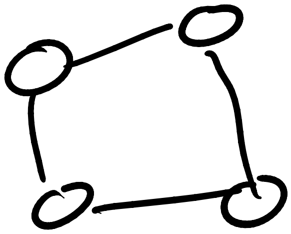
$\underline{m \geq n-1}$,

$P(k)$

$\underbrace{m \geq n-1}$

$m+1 \geq n-1$ obviously holds

P(k)

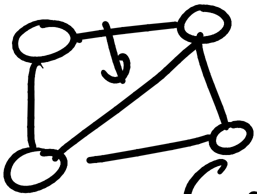


$V(G) = V(G')$

$n' = n$

$m' = m+1$

$m' \geq n'-1$ holds



P(k+1)

Case 4:

how does this affect n, m, n', m' as above?