LECTURE FOUR

Monday, January 24, 2022 10:19 AM

Proof techniques:

Induction

- weak: $P(1) \dots P(k), P(k+1)$

  $P(1)$: show; $P(k)$: assume via I.H.; $P(k+1)$: prove

  construction

  construction

- strong: $P(1) \dots P(k) \dots P(n)$

  $P(1)$: show; $P(k)$: show assumptions holds → I.H.; $P(n)$: assume

  prove

Necessity and sufficiency

aka equivalence relations

aka iff

aka $\Leftrightarrow$

A if and only if B

A iff B

$\boxed{A \Leftrightarrow B}$ → to prove

Show
$A \Rightarrow B$
$B \Rightarrow A$

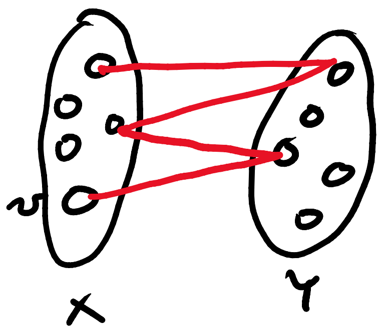
consider $G \in$ some class

P = some property

Necessity: if $G \Rightarrow P$
sufficiency: if $P \Rightarrow G$

G is bipartite iff G has no odd cycles

G is bipartite $\Rightarrow$ G has no odd cycles



- consider all possible paths from v

Note: any closed path must be even ✓
(any cycle must be even)

G has no odd cycles $\Rightarrow$ G is bipartite

- Assume G is connected
- consider some $v \in V(G)$

define W, Z vertex sets

define $f(u)$ = shortest u,v-path

$W = \{ \text{all } a \in V(G) : f(a) = \text{even} \}$

$Z = \{ \text{all } b \in V(G) : f(b) = \text{odd} \}$

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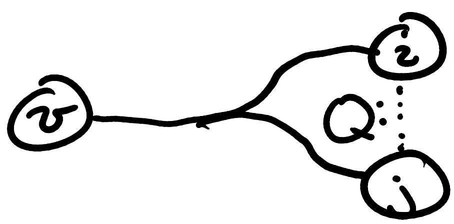
??? Q ?: are W, Z independent?

Consider shortest paths from v to two vertices $i, j \in W$ or Z



Note: $f(i), f(j)$ are both even or odd

★ parity argument ★

★ ★

→ any closed walk from $v \to i \to j \to v$ will be <u>odd</u>

From last class: all closed odd walks have an odd cycle

→ if $(i, j) \in E(G)$

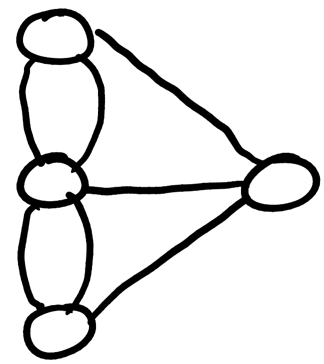
$\Rightarrow \exists\, C_n : n = \text{odd}$

✗ contradiction ✗

$\Rightarrow$ all vertices in W, Z are independent $\boxed{\checkmark}$

Recall: Euler and the Bridges of Königsberg

as a graph



Euler: Does a closed trail containing all edge exist?

aka Euler Tour
aka Euler Circuit

First: Prove if $\forall v \in V(G): d(v) \geq 2$
$\Rightarrow \exists C_n \in G$ for some n

To do this: <u>Extremal arguments</u>

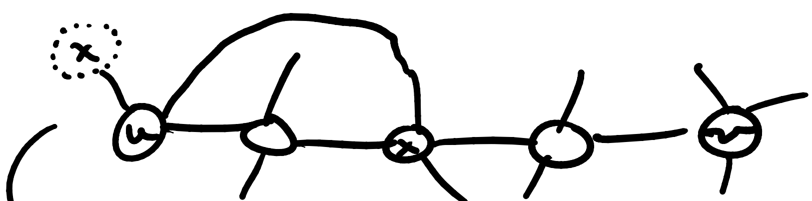
aka the extremal principle

$\exists$ within a set some subset

∃ within a set some subset that is maximal/maximum and minimal/minimum for some well-ordered property

$\forall v \in V(G): d(v) \geq 2 \Rightarrow \exists C_n \in G$

Consider: some maximum length path $P \in G$



$P \to u,v$-path

can x exist in this configuration?

NO

why? our selection of P

x,v-path would be longer

→ x has to be in P

=> closed path on u

→ cycle ☑

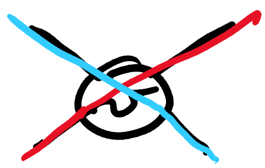
Eulerian graphs: graphs w/ Euler Tour

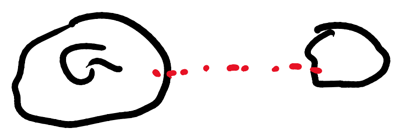
Eulerian graphs: graphs w/ Euler tour

G is Eulerian $\iff \forall v \in V(G): d(v) = \text{even}$ and at most 1 nontrivial component

G is Eulerian $\implies$ all $d(v)$ even and 1 nontrivial comp at most

G has Euler Tour. For every vertex we visit, we traverse one edge in and one edge out. $\implies$ all $d(v)$ even





Trivial to see why there can be at most 1 nontrivial component ✓

all $d(v)$ even at most 1 NTC $\implies$ G is Eulerian

(strong) Induction on $m = |E(G)|$

Basis: $P(m=0) \to \{\ \}$ trivial tour

$P(n)$: assume G has all even degrees and at most 1 NTC degrees and at most $1 N/C$

From our earlier proof: $\exists C \in G$

$P(k) = H = G - C$

Note 1: H has all even degrees

Note 2: H might be disconnected

But: assumptions hold for all components of H

→ we invoke our I.H. on all components of H

$\Rightarrow \exists$ Euler tours on each component

↳ How can we get back to $P(n)$?

Proof by ALGORITHM

To complete our proof, algorithmically combine the tour(s) on H with C to show a tour on G.

Our algorithm:

Our algorithm:

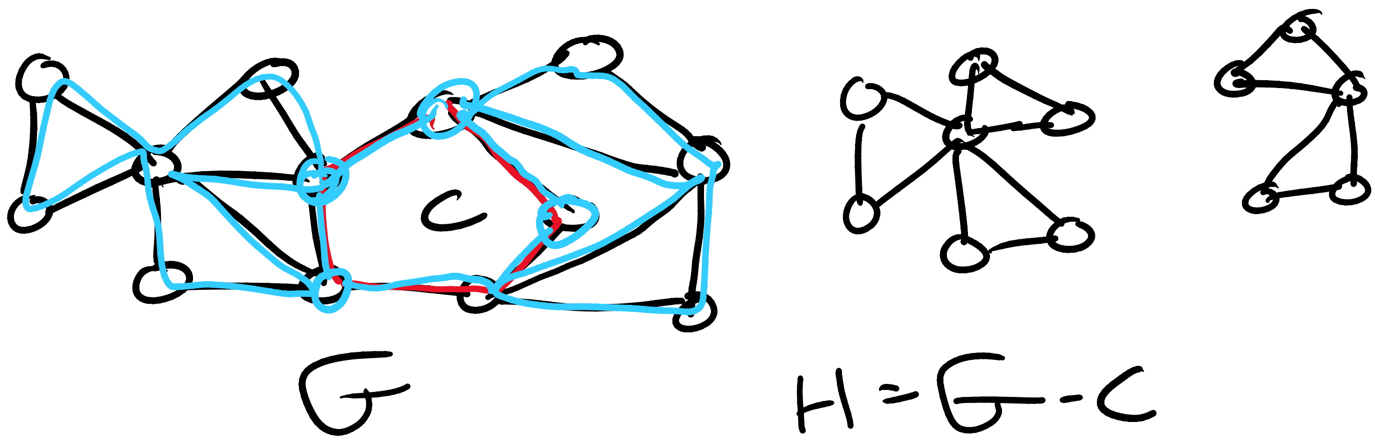
start on some $v \in C$

if $d(v) = 2$ continue on C

else $\exists$ a tour from v on a component of H (via I.H.)

continue on C to a new v

$\rightarrow$ Output: Euler Tour on G $\boxed{\checkmark}$



Degrees

recall $n = |V(G)|$, $m = |E(G)|$

degree of $v \in V(G) \rightarrow d(v)$

d_v

For graph G

maximum degree $\rightarrow \Delta(G)$

minimum degree $\rightarrow \delta(G)$

minimum degree $\to \delta(G)$

G is k-regular if

$$\Delta(G) = k = \delta(G)$$

aka $\forall v \in V(G): d(v) = k$

Note: all cycles are 2-regular
all cliques K_n are
$(n-1)$-regular

Degree sum formula:

$$\sum_{v \in V(G)} d(v) = 2m \to \text{even}$$

why: each edge adds +1 degree
to endpoint vertices