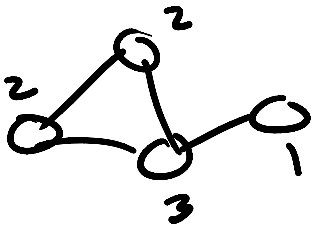
Lecture 5 - More degrees and directed graphs

Thursday, January 27, 2022 10:01 AM

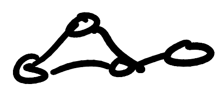
Degree sequence: list of all degrees for all vertices in same graph

G 

$S_G = \{1, 2, 2, 3\}$

Graphic sequence: a list of degrees that can <u>realize</u> a simple undirected graph

<u>realize</u>: a graph can be constructed with those degrees

$S_1 = \{1, 2, 2, 3\}$ 

S_1 is graphic

$S_2 = \{1, 2, 2, 2\}$

S_2 is <u>NOT</u> graphic

→ sum of degrees is odd

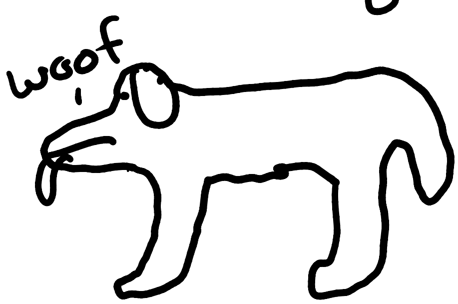
Note: even degree sum is <u>necessary</u> but not sufficient

$S_3 = \{10, 12\}$

S_3 not graphic

↳ Why?

How can we tell if a sequence is graphic or not?



Big Dawg of G.T.

Havel-Hakimi Theorem

consider sequence $S = \{d_1, d_2 \dots d_n\}$ where $d_1 \geq d_2 \geq \dots \geq d_n$ is graphic iff sequence

$$S' = \{d_2 - 1, d_3 - 1, \dots d_{(d_1+1)} - 1, \dots, d_n\}$$

is graphic

Example: $S = \{\cancel{3}, 2, 2, 1\}$ $\quad d_1 = 3$

$\phantom{S = \{3, } \overset{}{-1} \quad -1 \quad -1$

$S' = \{\cancel{1}, 1, 0\}$

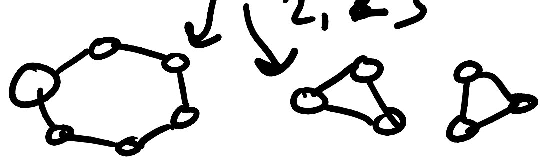
$\phantom{S' = \{} -1$

- d_2
- d_3
- $d_4 = d_{(d_1+1)} = d_{(3+1)} = d_4$

$S'' = \{0, 0\}$

↖ graphic

$S = \{2, 2, 2, 2, 2, 2\}$



Note: we can realize a graph of a given sequence using this processs

→ map each value in the sequence to a vertex

→ we draw an edge (d_i, d_j) when some d_j is decremented by the removal of d_i when going from $S \to S'$

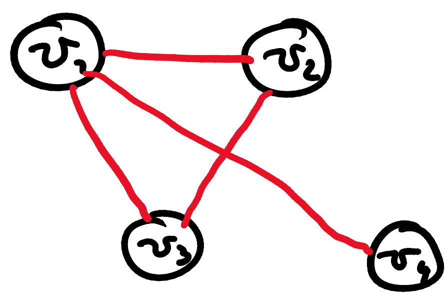
$$S = \{\overset{v_1}{\cancel{3}}, \overset{v_2}{2}, \overset{v_3}{2}, \overset{v_4}{1}\}$$

$$\phantom{S = \{3,}\ -1\ \ -1\ \ -1$$

$$S' = \{\overset{v_2}{\cancel{1}}, \overset{v_3}{1}, \overset{v_4}{0}\}$$

$$\phantom{S' = \{1,}\ -1$$

$$S'' = \{0, 0\}$$



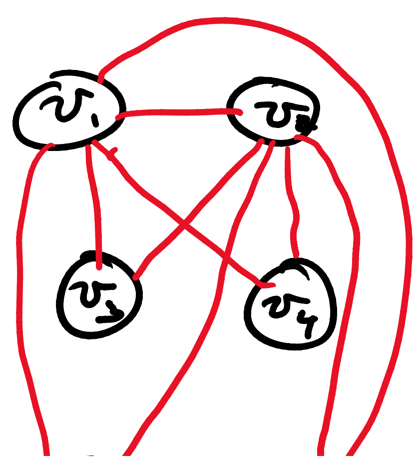
Example 2:

$$S = \{\overset{v_1}{\cancel{5}}, \overset{v_2}{5}, \overset{v_3}{2}, \overset{v_4}{2}, \overset{v_5}{2}, \overset{v_6}{2}\}$$

$$\phantom{S = \{5,}\ -1\ \ -1\ \ -1\ \ -1\ \ -1$$

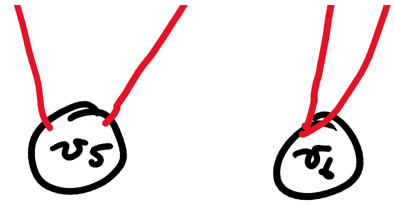
$$S' = \{\cancel{4}, 1, 1, 1, 1\}$$

$$\phantom{S' = \{4,}\ -1\ \ -1\ \ -1\ \ -1$$



$S' = \{\cancel{4}, 1, 1, 1, 1\}$

$-1 \quad -1 \quad -1 \quad -1$



$S'' = \{0, 0, 0, 0\}$

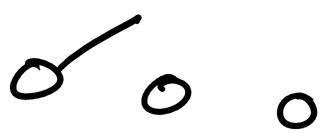
$S = \{\cancel{6}, 5, 4, 3, 2\} \rightarrow$ Not graphic

$-1 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1$

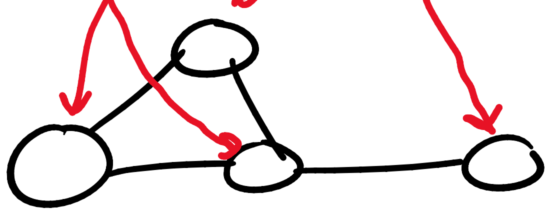
$S = \{\cancel{3}, 2, 1, 1\}$

$-1 \quad -1 \quad -1$

$S' = \{1, 0, 0\} \rightarrow$ NOT graphic

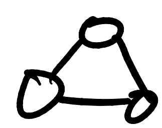


$S = \{3, 2, 2, 1\}$



$$\begin{array}{cccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & \\ S = \{ & 2, & 2, & 2, & 2, & 2, & 2 & \} \end{array}$$

$-1 \quad -1$



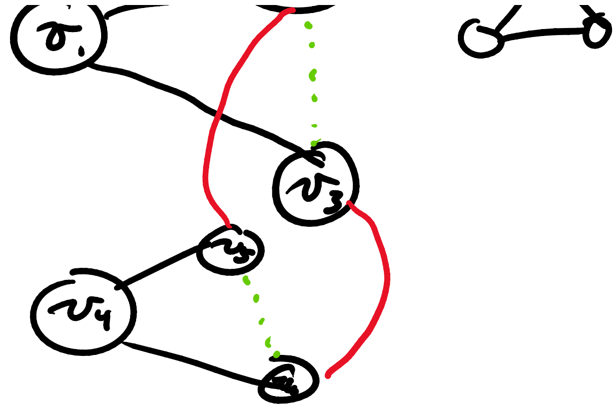
$$\begin{array}{ccccccc} & v_4 & v_5 & v_6 & v_2 & v_3 & \\ S' = \{ & \cancel{2} & 2 & 2 & 1 & 1 & \} \end{array}$$

$S' = \{\cancel{2}\ 2\ 2\ 1\ 1\}$

$-1 \quad -1$

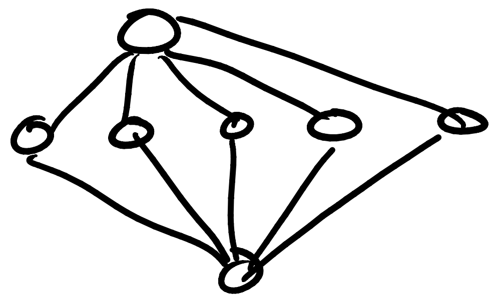
$v_5 \ v_6 \ v_2 \ v_3$

$\{1\ 1\ 1\ 1\}$



Braincercize: can all possible realizations for a given graphic sequence be constructed via the above process?

→ No



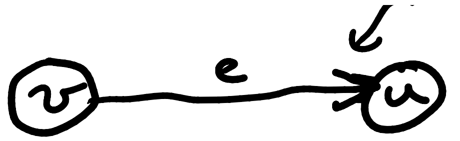
Directed graphs

aka digraphs

(also known as)

→ we consider directionality for every edge

$e = (v, u)$



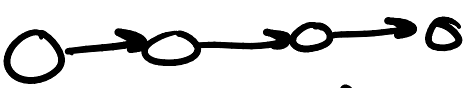
$e = (v, u)$

tail
predecessor

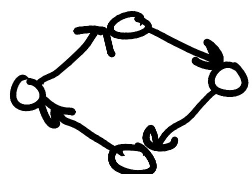
head
successor

walks, paths, trails, cycles

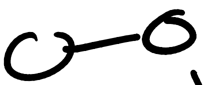
=> same definitions as with undirected graphs, but must follow edge direction



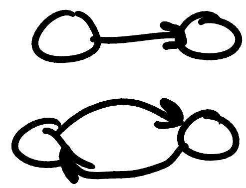
directed path



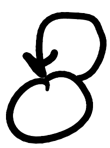
directed cycle



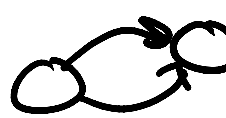
simple graph



simple digraphs



self loop



<u>not</u> simple

Adjacency matrix: no longer guaranteed to be symmetric

→ a nonzero at (i,j) implies an edge from $i \to j$

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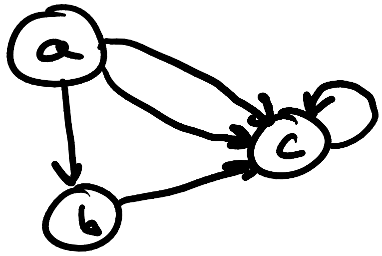
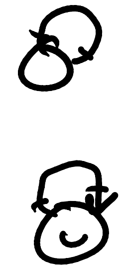


$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 2 \\ b & 0 & 0 & 1 \\ c & 0 & 0 & \boxed{1} \end{array}$$



Σ row = out degree

Σ column = in degree

out degree $d^+(v)$ = # edges coming from v

in degree $d^-(v)$ = # edges going into v

$N^+(v)$ = out neighborhood

$N^-(v)$ = in neighborhood

$\delta^+(G), \delta^-(G)$ = min out, in degrees in G

$\Delta^+(G), \Delta^-(G)$ = max out, in degrees in G

Degree sum formula

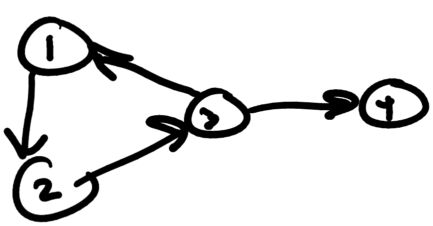
Degree sum formula

$$\underbrace{\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = m}$$

why: each edge contributes exactly +1 to the sums of in and out degrees

Degree sequence of directed graphs

- We must now consider (in, out) degree pairs



$S = \{(1,1), (1,1), (1,2), (1,0)\}$

How can we tell if S is (di)graphic?

Necessary condition:

sum of in degrees must equal sum of out degrees

Sufficient condition:

Sufficient condition:

if $\underbrace{\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v)}_{\text{over } S} \Rightarrow S$ is graphic

Note: for loopy multi-digraphs

Proof by ALGORITHM

consider $(d_i^+, d_i^-) : 1 \le i \le n$

$$m = \sum d^+ = \sum d^-$$

consider m lines

d_i^+ dots get labeled i

d_i^- dots get labeled $-i$

consider n vertices $1 \dots n$

$v_1 \dots v_n$

For each line

construct an edge from the positive label to the negative label via vertex set

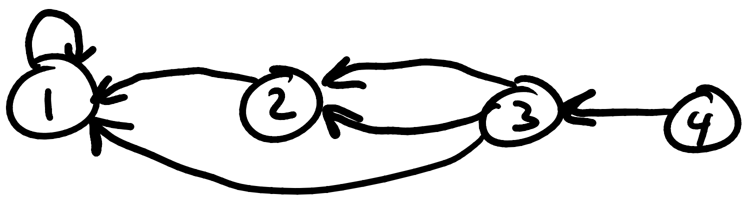
$i=1 \quad 2 \quad 3 \quad 4$

$(1,3)\ (1,2)\ (3,1),(1,0)$

$$\begin{array}{cccc} z=1 & 2 & 3 & 4 \\ (1,3), & (1,2), & (3,1), & (1,0) \end{array}$$

out in

$$\begin{array}{cccccc} 1 & 2 & 3 & 3 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ -1 & -1 & -1 & -2 & -2 & -3 \end{array}$$



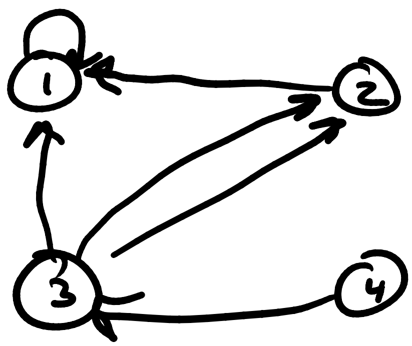
Eulerian digraph: a digraph is Eulerian iff there exists a closed trail containing all edges

→ proof = same as with undirected

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ (\underline{1},\underline{3}) & (\underline{1},2) & (\underline{3},1) & (\underline{1},0) \end{array}$$

$$m = \sum d^- = \sum d^+ = 6$$

$$\begin{array}{cccccc} 1 & 1 & 3 & 3 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 1 & 2 & 2 & 3 \end{array}$$


1
2
3
4