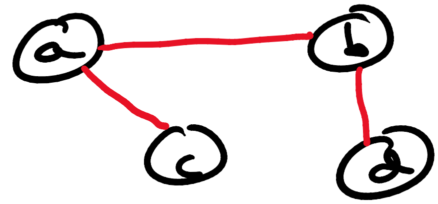
Lecture 6 - Trees

Monday, January 31, 2022 9:59 AM

Havel-Hakimi:



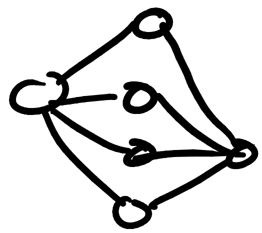
$S = \{\overset{a}{\cancel{2}}, \overset{b}{2}, \overset{c}{1}, 1^{d}\}$
$\quad\quad\quad\ \ -1 \quad -1$

$S' = \{\overset{b}{1}, \overset{c}{0}, \overset{d}{1}\}$ $\updownarrow$ S is graphic iff S' is graphic

$S' = \{\overset{b}{\cancel{1}}, \overset{d}{1}, \overset{c}{0}\}$
$\quad\quad\quad -1$

Q: can H-H realize all possible simple configurations?

A: No



$S = \{4, 4, 2, 2, 2, 2\}$

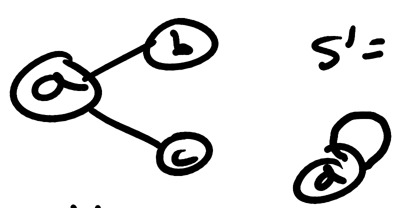
by counterexample

$S = \{4, 2, 2, 2, 2, 4\}$

Q: Can we *arbitrarily* permute degrees to generate all configurations?

A: No consider $S = \{2\ 2\ 1\ 1\}$

$S = \{\overset{a}{2}\ \overset{b}{1}\ \overset{c}{1}\ \overset{d}{2}\}$



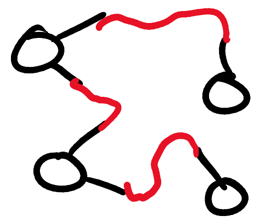
$S' = \{2, 0, 0\}$

Note: this can generate all loopy multigraphs

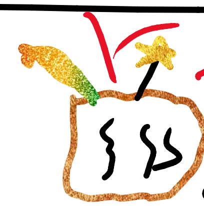
... on loopy multigraphs

↑ configuration model

$S = \{2211\}$



Bag o' tricks



aka proof techniques

Structural arguments

→ consider v of degree 2 configured in X way

Extremal arguments

→ consider maximum path P, cycle C, etc

Parity arguments

→ even + even = even, odd + odd = even
even + odd = odd

Weak induction

→ $P(1) \ldots . \; P(k), P(k+1)$

basis (for $P(1)$), assume (for $P(k)$), construct (from $P(k)$ to $P(k+1)$), show (for $P(k+1)$)

strong induction construct

strong induction

→ P(1) ... P(k) ... P(n)

construct

show

basis   assume

Necessity and sufficiency

→ to prove equivalence ⇔

first show ⇒, then show ⇐

Proof by algorithm

→ here's an algorithm than can guarantee we have X

Proof by counter-example

→ can we guarantee X: no, here's a counter-example

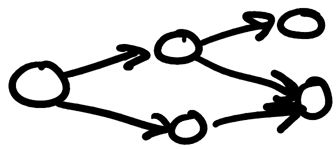
Trees

Tree - a connected undirected simple acyclic graph

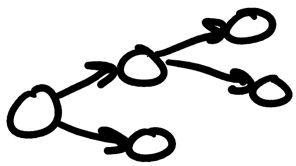
acyclic: contains no cycles

Forest - a disconnected undirected simple acyclic graph simple acyclic graph

DAG - directed acyclic graph



Polytree - a DAG where the underlying graph is a tree



underlying: undirected graph created by removing directionality

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Tree T necessary conditions

Obviously: simple undirected connected acyclic

T is minimally connected
→ removing any edge will disconnect T

T is maximally acyclic
→ adding any edge will create a cycle

T has $|E(T)| = |V(T)| - 1$

T has a single u,v-path $\forall u,v \in V(T)$

T is bipartite

$\forall u, v \in V(T)$

T is bipartite

Think about: which of the above is also sufficient?

Prove T is a tree $\to$ T is bipartite

$\to$ use weak induction on $n = |E(T)|$

($n = |V(T)|$)

Basis $P(m=1) \to$ o—o ✓

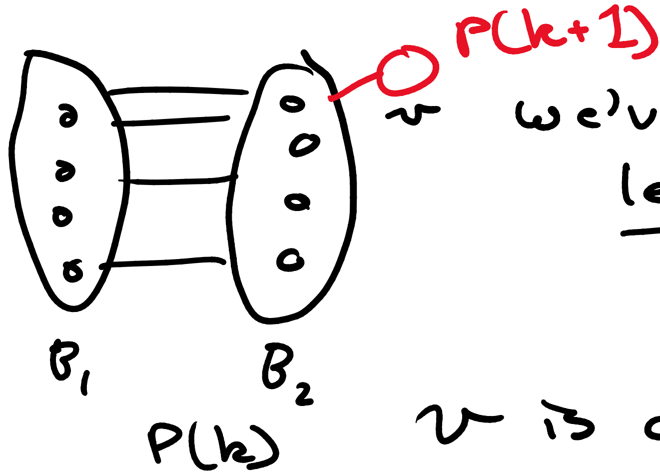
$P(n=1) \to$ o ✓ $\leftarrow P(m=0)$

assume via I.H. P(k) is bipartite

$\forall P(k)$ case $k = |E(T)|$ or $k = |V(T)|$

construct $\hookrightarrow P(k+1)$ add an edge and a new vertex

We've assumed P(k) is bipartite



we've added a leaf vertex v

$\leftarrow$ degree-1 vertex in a tree

v is connected to some u w.l.o.g. is in set B_2

$\to$ we can put v in B_1 and both

$\rightarrow$ we can put v in B_1 and both B_1, B_2 are still independent

$\Rightarrow P(k+1)$ is bipartite $\boxed{\checkmark}$

Prove if T is a tree $\rightarrow \forall u,v \in V(T): \exists P$

P = unique u,v-path

strong induction on $m = |E(T)|$

Basis: $P(1)$: $\underset{u}{\circ}\!-\!\underset{v}{\circ}$ looks unique ✓

consider $P(n)$, which is a Tree

$\hookrightarrow$ construct $P(k)$ s.t. $P(k)$ is a tree

Note: all trees have leaves

$\rightarrow$ we can remove a leaf + edge (xy) to get to $P(k)$, a tree

since initial assumption (of treeness) holds $\rightarrow$ we can invoke I.H. on $P(k)$

$\hookrightarrow \forall u,v \in V(P(k)): \exists P$

P = unique u,v-path

Let's bring it on back
to $P(n)$ case

→ we add back that leaf vertex
x and edge (x,y)

Let's bring it on home

→ show our result via I.H. on $P(k)$
still holds on $P(n)$

we know on $P(n)$

$\forall u,v \in \{V(P(n)) - x\} : \exists P$, unique u,v-path

To get unique u,x-paths $\forall u \in \{V(P(n)) - x\}$

→ we simply consider unique u,y-paths
and add edge (x,y) ☑

Extra fun definitions

distance - $d(u,v)$ = length of the
shortest u,v-path

diameter - $D(G)$ = length of the largest
shortest u,v-path $\forall u,v \in G$

$$= \max_{\forall u,v \in V(G)} d(u,v)$$

lil' epsilon

lil' epsilon

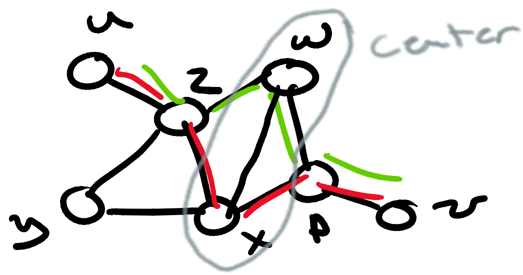
$$= \max_{\forall u,v \in V(G)} d(u,v)$$

Eccentricity $-$ $\epsilon(v) = \max_{\forall u \in V(G)} d(u,v)$

radius $-$ $R(G) = \min_{\forall v \in V(G)} \epsilon(v)$

Center of G = the induced subgraph of vertices with minimum eccentricity in G

G



$d(u,v) = 4$

$D(G) = 4$

$\epsilon(w) = 2$

$R(G) = 2$

Center of G