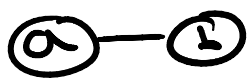
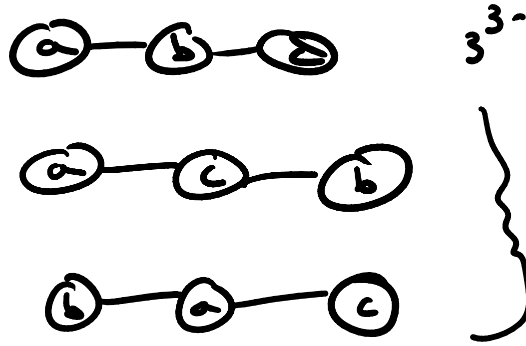
Cayley's formula: there exists $n^{(n-2)}$ possible trees for $|V(T)| = n$



$2^{2-2} = 1$



$3^{3-2} = 3$

think automorphism

Prüfer code: a sequence of labels for tree T s.t. the length of the sequence is $n-2$ and T's vertex labels comprise the sequence

$$A = \{a_1\ a_2 \ldots a_{n-2}\}$$

$S = \{$vertex labels of T$\}$ ← sortable

$a_i \in S$

How do we construct Prüfer code A?

CreatePrüfer(T):

$A = \emptyset$

CreatePrufer(T):

$A = \emptyset$

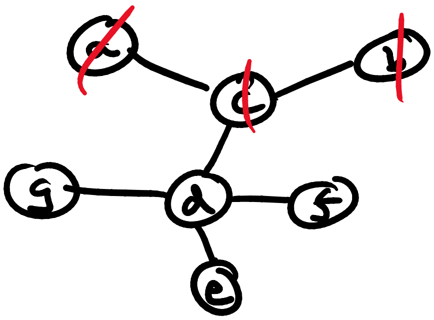
for $i = 1 \ldots (n-2)$:

ℓ = label of least remaining leaf

$T = T - \ell$

a_i = remaining neighbor of ℓ

$A \leftarrow a_i$



$S = \{\cancel{a}, \cancel{b}, \cancel{c}, d, \cancel{e}, \cancel{f}, g\}$

$A = \{c, c, d, d, d\}$

↑ Prüfer code of (T, S)

CreateTree(A, S):

$V(T) = S$

$E(T) = \emptyset$

consider all S as "unmarked"

for $i = 1 \ldots (n-2)$:

x = least unmarked in S that is not in $a_i \ldots a_{n-2}$

mark x in S

$E(T) \leftarrow (x, a_i)$

(x, y) = remaining unmarked vertices in S

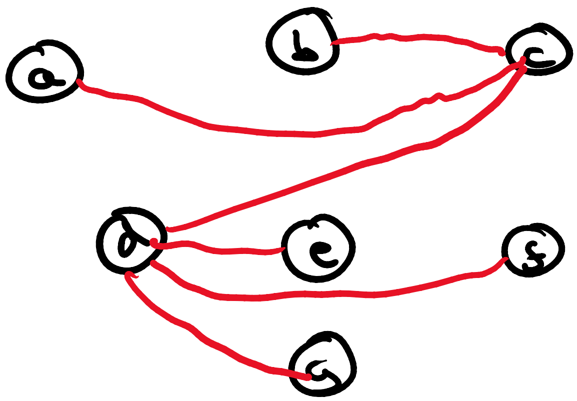
— remaining unmarked vertices in S

$$E(T) \leftarrow (x, y)$$

$a_i \ldots 1$

$$A = \{c, c, d, d, d\}$$

$$S = \{\cancel{a}, \cancel{b}, \cancel{c}, d, e, f, g\}$$



Takeaway → for a given tree T and vertex set S, we define a unique code A
and given code A and set S, we can construct a unique tree T

$$f(T) = A$$

$\leftrightarrow$ bijection

Bring it on back to Cayley

$\exists\, n^{n-2}$ possible trees

Why: There n^{n-2} ways to write a Prüfer code $A = \{a_1, \dots a_{n-2}\}$

Proofer code Prüf (Cayley's)

For sequence S of vertex labels where $|S| = n \Rightarrow \exists\, n^{n-2}$ possible Trees where $S = V(T)$

Note: we're going to prove the uniqueness and existence of Prüfer code $f(T) = A$ mapping

We'll use strong induction on $n = |S|$

Basis $P(2)$ $\overset{a}{o}-\overset{b}{o}$ $S = \{a, b\}$

$f(T) = A = \{\}$ ✓

$2^{2-2} = 2^0 = 1$

Consider $P(n) = T$

- tree T with $V(T) = S$, $|S| = n$
- consider x as least element of S where x is a leaf in T
- consider a to be neighbor

- consider a to be neighbor of x

Now consider our $P(k)$

From $P(n) \to$ remove $x \to P(k)$

$P(k) = T' = T - x$

$S' = S - x$

$A' = \{a_2 \ldots a_{n-2}\}$

via I.H., A' exists and is unique to given (S', T')

From $P(k) \xrightarrow{\text{add } x} P(n)$

- going from $A' \to A$, $T' \to T$, $S' \to S$

From $S' \to S$, we push x to the front of S'

From $T' \to T$ we add back edge (x, a)

From $A' \to A$

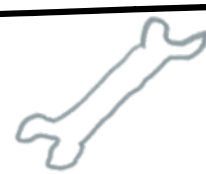
$\Rightarrow$ From our Prüfer code algorithm, the vertex x and edge (x, a) would be first selected for removal first selected for removal

$\Rightarrow$ so first value in A is guaranteed to be a

$A = \{a, \{A'\}\} \quad f(T) = A$

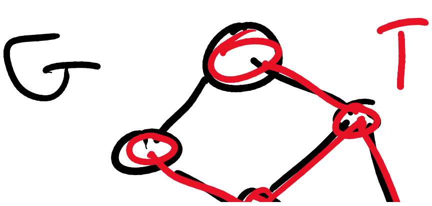
$\rightarrow$ so there exist a unique $f(T) = A$ for given S

$\Rightarrow \exists\, n^{n-2}$ possible automorphic tree configurations for $|V(T)| = n$, $|S| = n$ ☑

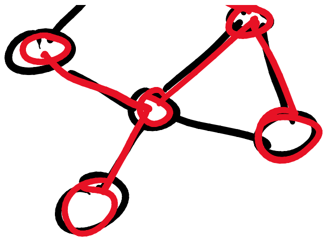
Spanning trees  

Spanning tree is a tree subgraph T of some graph G s.t. $V(T) = V(G)$

$\rightarrow$ spanning trees are acyclic connected subgraphs containing all vertices in the original graph



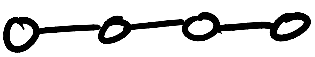
Note: the number of spanning trees of a



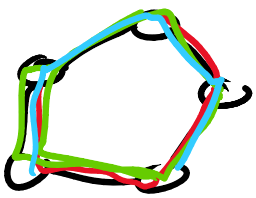
spanning trees of a complete graph is n^{n-2}

$\tau(G)$ = # spanning trees on G

$\tau(K_n) = n^{n-2}$



$\tau(P_n) = 1 \qquad \tau(T) = 1$



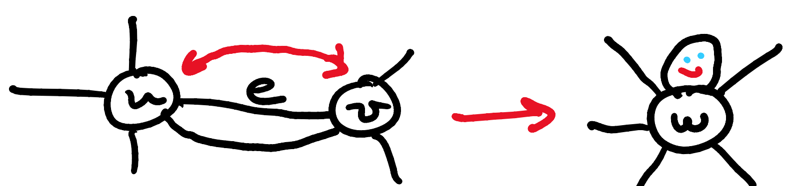
$\tau(C_n) = n$

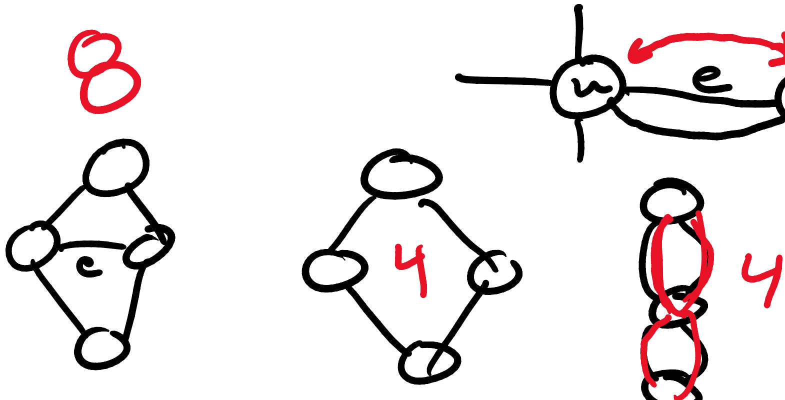
How can we count spanning trees in general?

Let's define a recurrence (sp?):

$$\underbrace{\tau(G)}_{\#\text{S.T.s}} = \underbrace{\tau(G-e)}_{\#\text{S.T.s w/o } e} + \underbrace{\tau(G\cdot e)}_{\#\text{S.T.s w/ } e}$$

↑ edge contraction

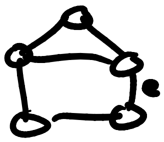
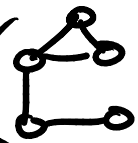
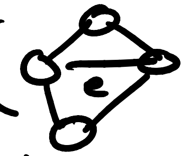


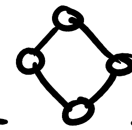
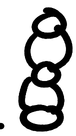


$G \qquad G-e \qquad G\cdot e$

$$\tau(G) = \tau(G-e) + \tau(G\cdot e)$$

$$\tau(G) = \tau(G - e) + \tau(G \cdot e)$$

$\tau\Big($  $\Big) = \tau\Big($  $\Big) + \tau\Big($  $\Big)$

$= 3 + \tau\Big($  $\Big) + \tau\Big($  $\Big)$

$= 3 + 4 + 4 = 11$

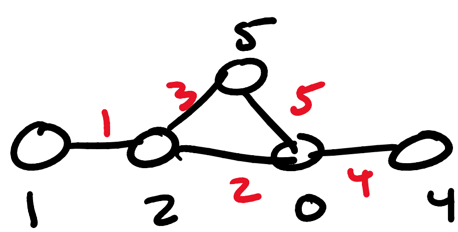
Graceful graphs

↳ graphs with a graceful labeling

Graceful labeling: a labeling of vertices and edges of G s.t.

$\forall v \in V(G)$: $L(v) = 0 \ldots m = |E(G)|$ and is unique

$\forall e = (u, v) \in E(G)$: $L(e) = |L(u) - L(v)|$ is unique



Graceful tree conjecture

(Ringel-Kotzig)

Graceful Tree Conjecture
(Ringel-Kotzig)

All trees are graceful
(unproven)