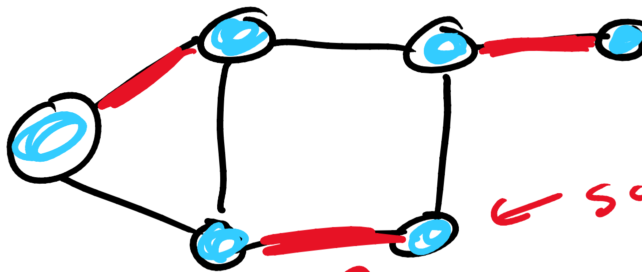
Lecture 11 - General Graph Matching and Covers

Thursday, February 17, 2022 7:34 AM

Review:

Match: set of edges w/o shared endpoints



← unsaturated

← saturated vertex

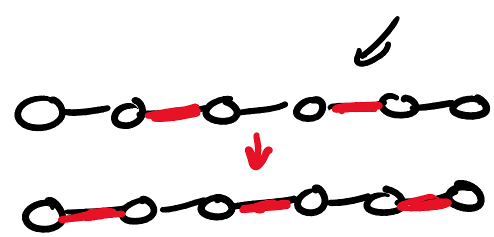
↑ matched edge

Maximum: largest possible match

Maximal: can't be made larger

Perfect: saturates all vertices

M-alternating path: 

M-augmenting path: 

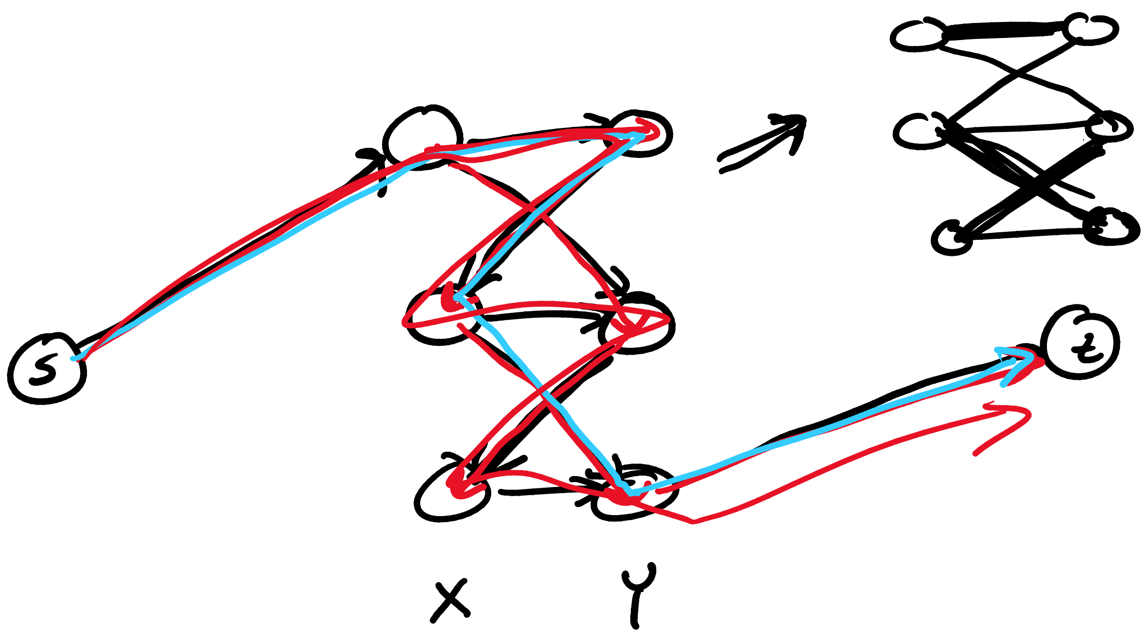
Berge: M is maximum on G iff G has no M-aug paths

Symmetric difference: XOR

↪ for matches: cycles or paths * relevant for quiz

Hall: $\exists M$ that saturate X in XY-bigraph $|X| \le |Y|$ iff $\forall S \subseteq X$ $|N(S)| \ge |S|$

Max match alg for bigraphs

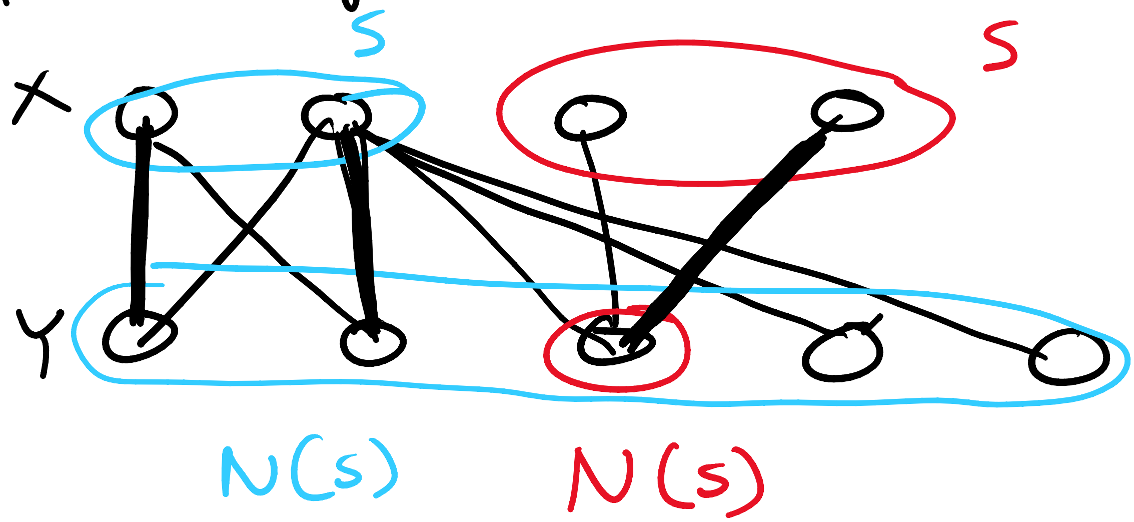


Contrapositive

$Q \to P \quad \neg P \to \neg Q$

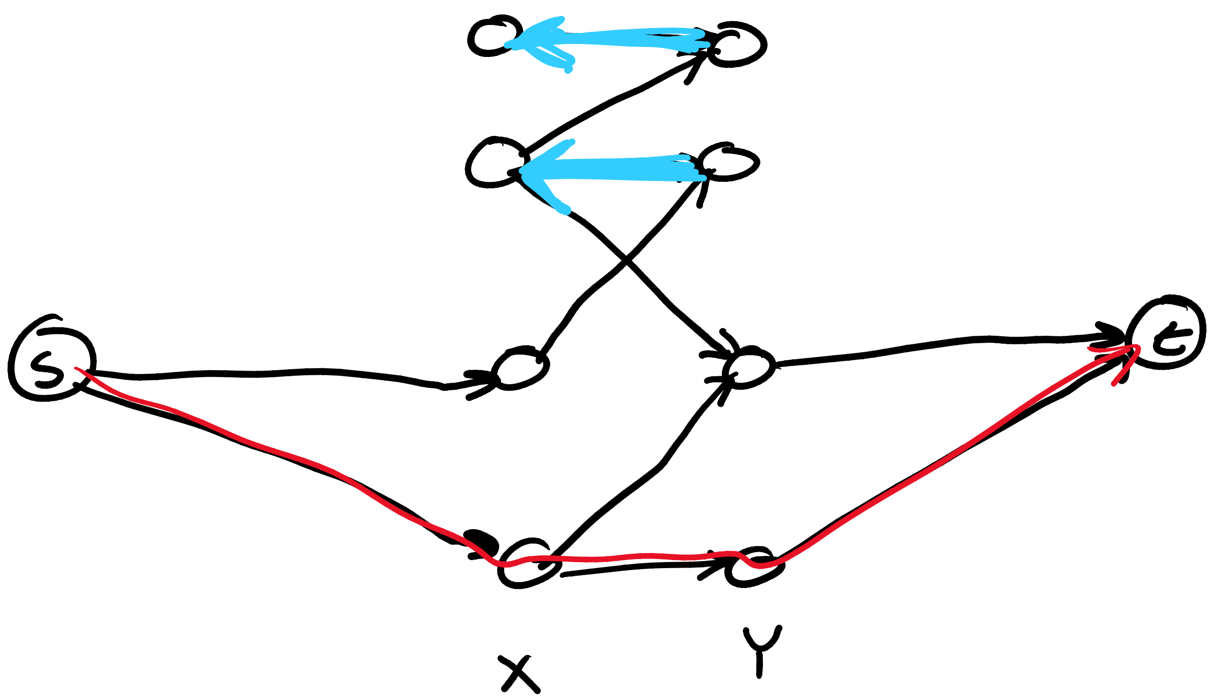
$Q \leftrightarrow P \qquad \neg Q \leftrightarrow \neg P$

Hall Example



$|S| = 2$ $\quad$ $|S| = 2$ $\quad |N(S)| = 1$

$|N(S)| = 5$ $\quad$ $\forall S \subseteq X \quad |N(S)| \geq |S|$



General graph matching

General graph matching

$o(G)$ = # odd components of G

Tutte's Theorem:

G has a perfect match (P.M.) iff $\forall S \subseteq V(G)$:

$$o(G-S) \le |S|$$

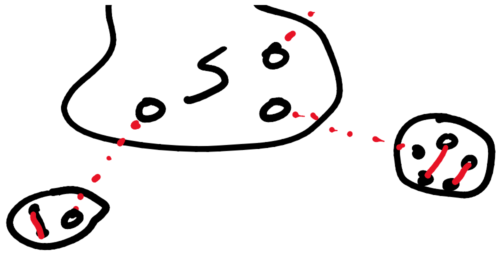
G has P.M. $\Rightarrow \forall S \subseteq V(G): o(G-S) \le |S|$

- consider some arbitrary S
- consider $G-S$
- Note: each odd component cannot be perfectly matched

→ at least one vertex in each odd component must match some vertex in S

$o(G-S) = 3$

$|S| \ge 3$



$|S| \geq 3$

$\Rightarrow o(G-S) \leq |S|$

$\forall S \subseteq V(G): o(G-S) \leq |S|$

$\Rightarrow G$ has a P.M.

★ contrapositive ★

$\underbrace{G \text{ has no P.M.}}$ $\Rightarrow \exists S$ s.t. $|S| < o(G-S)$

Note: condition holds if we add edges to G

Extreme

we consider an extremal choice of G', where G' is an edge-maximal graph with no P.M.

$G' + e$ has P.M.

Define $S = \forall v \in V(G): d(v) = n-1$

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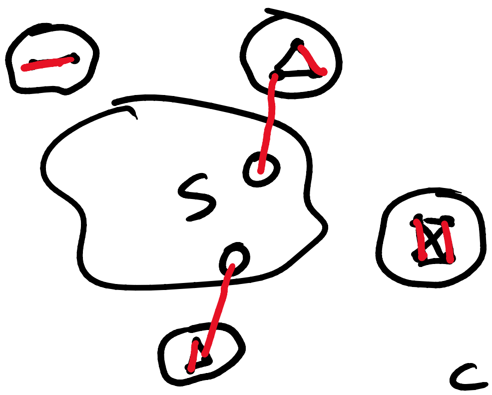
attached to all other $u \in V(G)$

Case 1: $G' - S \rightarrow$ all components are cliques



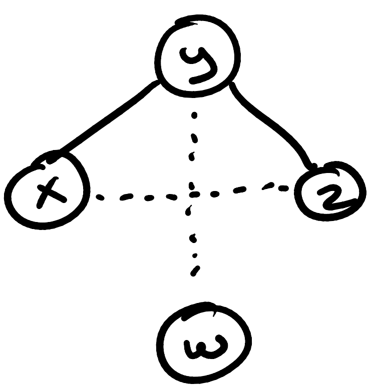
S must be "bad"

$|S| < o(G-S)$

otherwise we can construct a P.M.

contradiction

Case 2: $G' - S$ not comprised of cliques



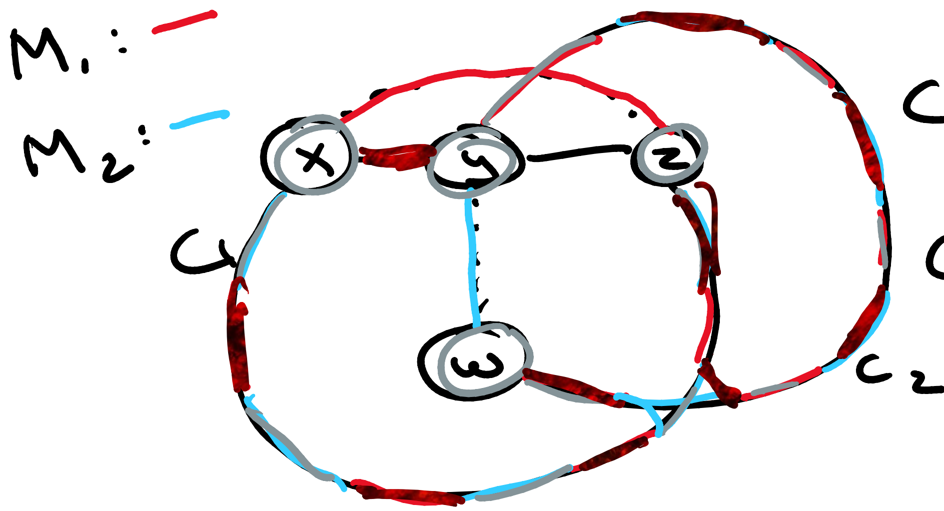
$\exists\, x, z$ s.t. $(x,z) \notin E(G'-S)$

$\exists\, y$ s.t. $(x,y)(z,y) \in E(G'-S)$

$\exists\, w$ s.t. $(y,w) \notin E(G'-S)$

show: adding edge $e(x,z)$ or $e=(y,w)$ creates a P.M. on $G'+e \Rightarrow \exists$ P.M. on G'

- define:
  - M_1 = P.M. on $G' + (x,z)$
  - M_2 = P.M. on $G' + (y,w)$
  - $F = M_1 \Delta M_2$ → must be paths or <u>cycles</u>



C_1 = cycle with (x,z)

C_2 = cycle with (y,w)

Case 2a: $C_1 \neq C_2$

P.M. on G' = all $e \in M_2$, $e \in C_1$
all other $e \in M_1$

→ P.M. w/o (x,z) or (y,w)

contradiction

⇒ S must be bad

Case 2b: $C_1 = C_2$

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P.M. on $G' = M_1$ on C_2 from w until x or z

if we reach x:

- P.M. on G' += $(x, y) + M_2$ from y to z

if we reach z:

- P.M. on G' += $(y, z) + M_2$ from y to x

$\Rightarrow$ either way, we have a P.M. w/o (x, z) or (y, w)

$\Rightarrow$ Contradiction

so case 2 can't exist

so S must be bad

$|S| < o(G-S)$

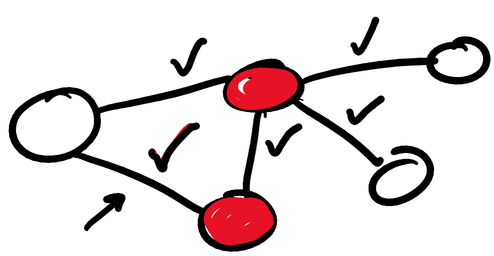
$|S| < o(G-S)$

in all cases $\square$

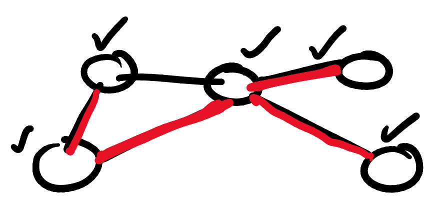
$$\boxed{\text{Tutte: } G \text{ has P.M.} \iff \forall S \subseteq V(G): \; o(G-S) \le |S|}$$

Vertex cover: a set $Q \subseteq V(G)$ that has at least on endpoint for all $e \in E(G)$

Edge cover: a set $L \subseteq E(G)$ that has at least one edge incident on all $v \in V(G)$



vertex cover



edge cover

König-Egerváry: on bipartite graph $G \Rightarrow$ size of a minimum graph $G \Rightarrow$ size of a minimum vertex cover = size of a maximum match

$|M_{max}|$ = max match $\rightarrow |M_{max}| = |C_{min}|$

$|C_{min}|$ = min cover

Note: $|C| \geq |M|$ for any cover and match

$\rightarrow$ every matched edge needs to be covered by one $v \in C$