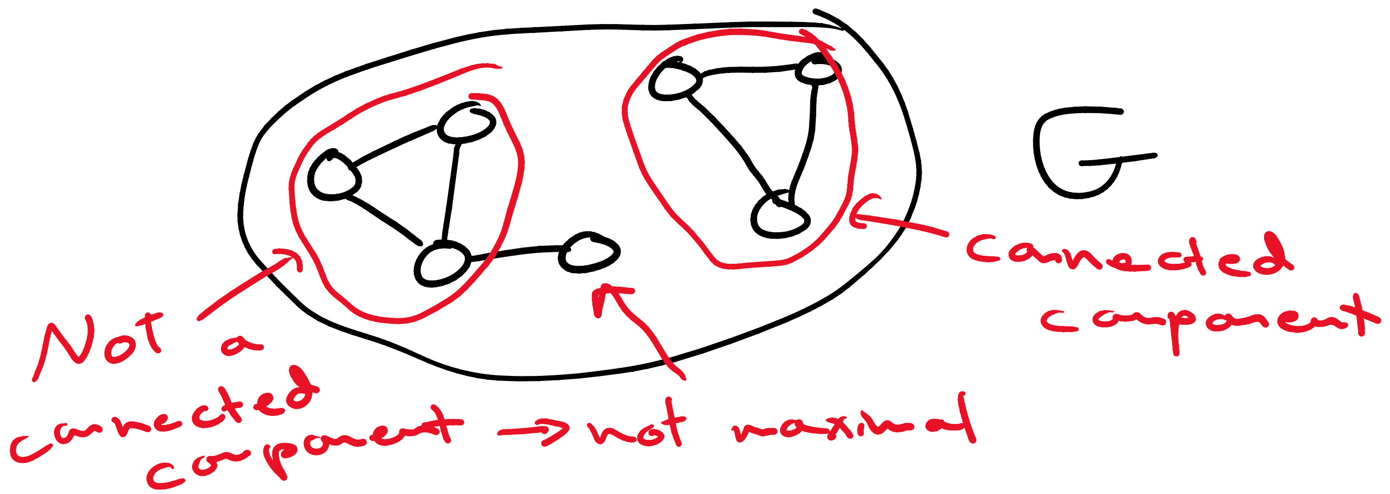
Lecture 12 - Connectivity

Monday, February 21, 2022 7:57 PM

Review (quick) of connectivity

Graph G is connected if $\forall u,v \in V(G)$
$\exists\, u,v$-path

connected component is a maximal
connected subgraph of some G



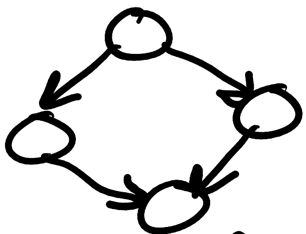
Directed Graphs

weak and strong connectivity

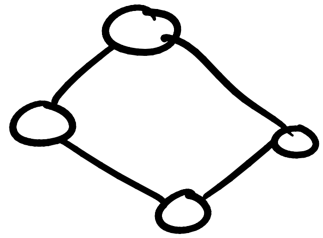
weak → a digraph is weakly connected if the underlying graph is connected

connected

graph if we ignore the edge directions (opposite of orientation)



G

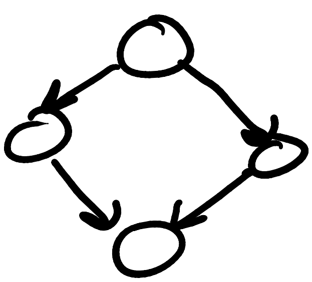


underlying graph

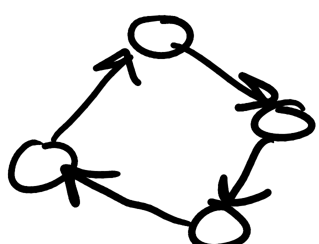
weakly connected, as the underlying graph is connected

strong → a digraph G is strongly connected if $\forall u,v \in V(G)$ $\exists$ u,v-path

directed path



✗ NOT ✗ strongly connected

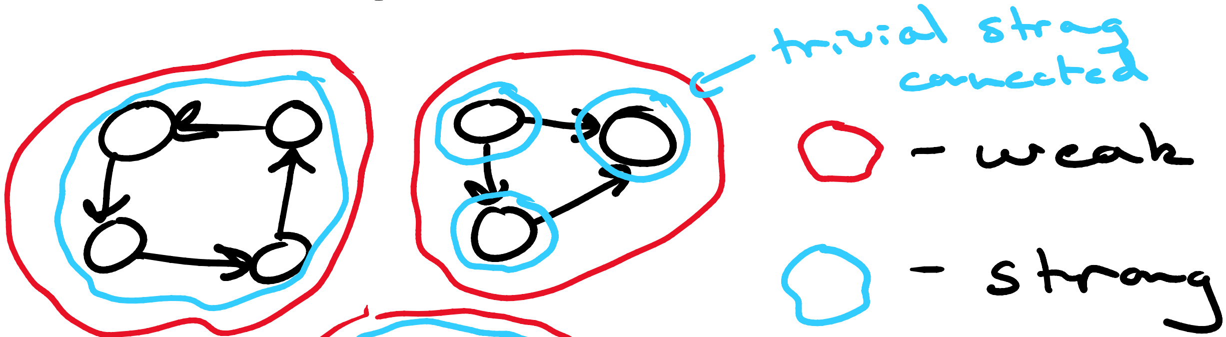


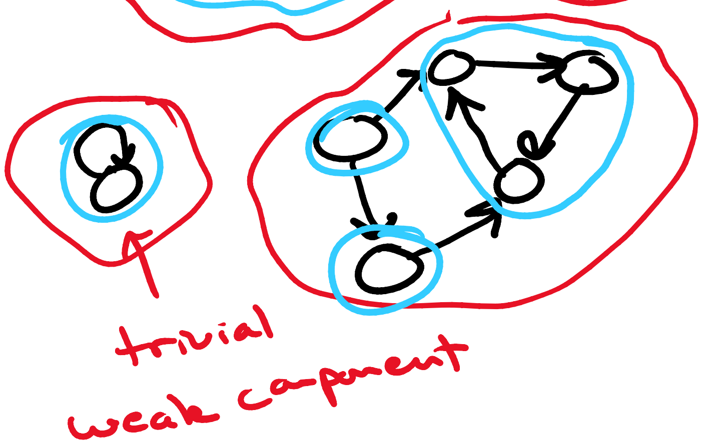
✓ Yes strongly connected ✓

<u>weakly connected component</u> – a maximal

weakly connected component - a maximal weakly conected subgraph

strongly connected component - a maximal strongly connected subgraph





connectivity decomposition - identifying all components within a graph

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Vertex Connectivity

Cut vertex - a vertex v in some G s.t. $G-v$ has more connected components than G

Separating set - a subset $S \subseteq V(G)$ on a graph G s.t. $G-S$

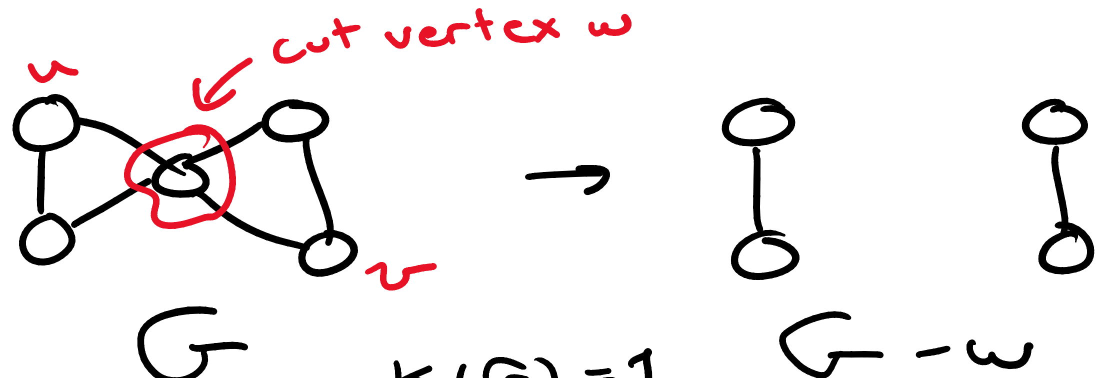
on a graph G s.t. $G - S$ has more components than G or $G - S$ only has a single vertex

AKA vertex cut

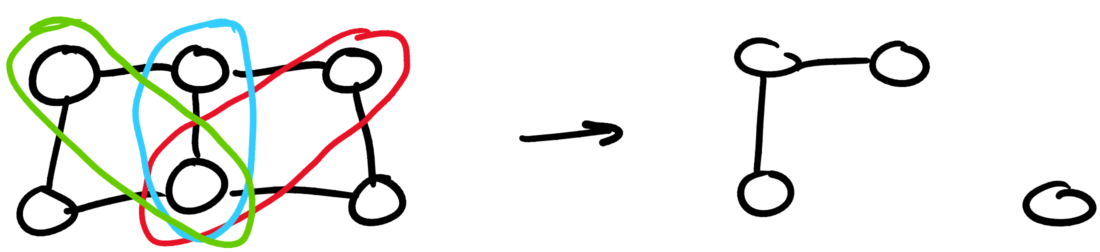
vertex separator

Connectivity of $G = \kappa(G) = k$ is the size of a minimum vertex separator

↳ G is k-connected if $k = \kappa(G)$



$\kappa(G) = 1$



$\kappa(G) = 2$

Note: for connectivity, the maximum size of a separator

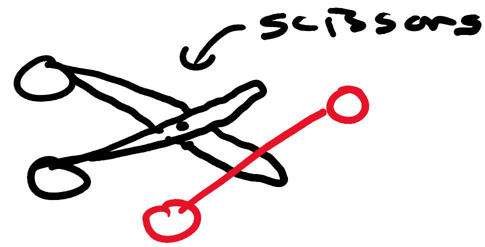
$\geq |V(G)| - 1$

maximum size of a separator
is $|V(G)| - 1$

↳ a clique K_n is $(n-1)$-connected

Edge Connectivity

cut edge - an edge e in some G
s.t. $G - e$ has more components
than G



Disconnecting set - a set of edges
subset → $F \subseteq E(G)$ s.t. $G - F$ has more
components than G

AKA edge cut

Bond: minimal edge cut
↳ not necessarily minimum

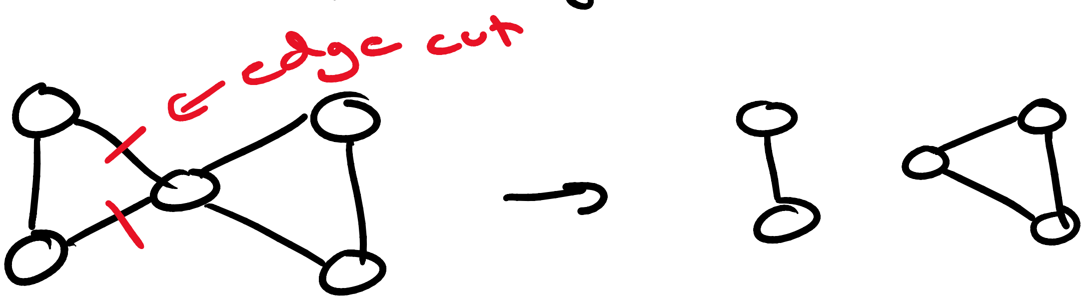
<u>edge-connectivity</u> of $G = \kappa'(G) = k$

minimum size of an edge cut

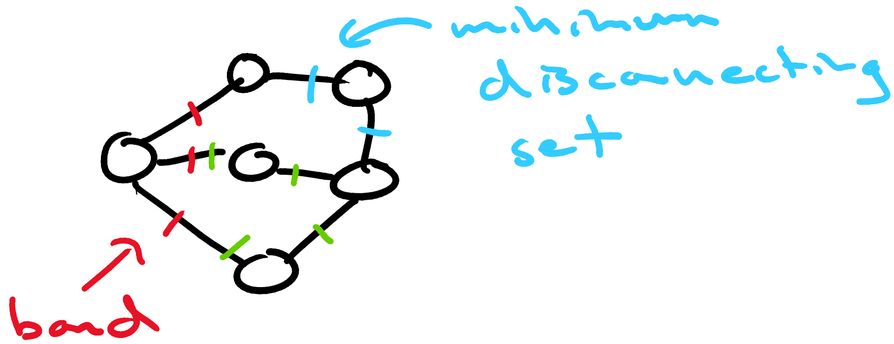
minimum size of an edge cut

→ G is k-edge-connected

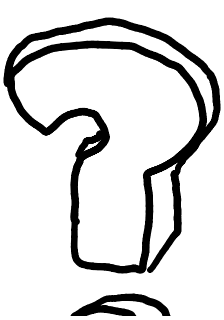
Note: if G is k-connected or k-edge-connected then G is (k-1)-connected and (k-1)-edge-connected



$K'(G) = 2$

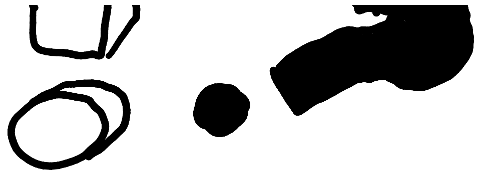


Bounds on connectivity



$K(G) \overset{?}{\underline{<>}} K'(G) \overset{?}{\underline{<>}} \delta(G)$

can we place bounds on these relationships



these relationships

We saw: $\kappa'(G) \le \delta(G)$

Why? Trivially, removing all edges incident on a minimum degree vertex will disconnect that vertex

Likewise: $\kappa(G) \le \delta(G)$

Why? Can remove all neighbors of our min. degree vertex to disconnect it

But what about $\kappa'(G)$ and $\kappa(G)$?

EXXXXTEMAL ARGUMENT

- consider a minimum edge cut F that separates G into $S, \bar{S}$ s.t.

that separates G into $S, \bar{S}$ s.t.

$\bar{S} = G - S$ and $S, \bar{S} \subseteq V(G)$

Case 1: $\forall u \in S, \forall v \in \bar{S}: \exists (u,v) \in E(G)$

$\rightarrow$ we know

$$K'(G) = |F| = |S||\bar{S}| \geq |V(G)| - 1$$

$\updownarrow$

Note: $K(G) \leq |V(G)| - 1$

$$K'(G) \geq K(G)$$

Case 2: $\exists x \in S, \exists y \in \bar{S} : (x,y) \notin E(G)$

STRUCTURAL ARGUMENT

define $T =$ all $u \in N(x): u \in \bar{S}$

and all $v \in S - x: \exists (v,z) \in E(G)$
$z \in \bar{S}$

Note: all x,y-paths must go through some vertex in T

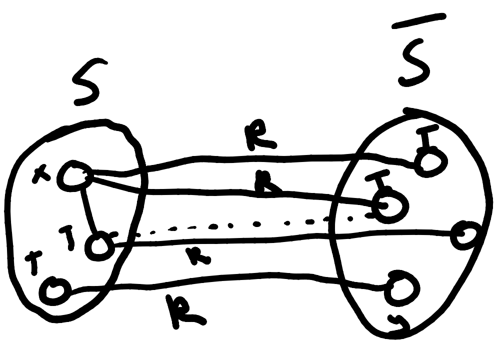
some vertex in 1

$\hookrightarrow$ T is a x,y-vertex cut

define R = all $e \in (x, w)$: $w \in T \cap \bar{S}$

and $f = (a, b)$: $a \in T \cap S$, $b \in \bar{S}$

(one of each possible for each a)



$\hookrightarrow$ Note: $|R| = |T|$

As we've selected only one f for R out of multiple possible for each b

$\hookrightarrow$ $|R| \leq |F|$

$\Rightarrow |T| \leq |F|$

$\Rightarrow \boxed{K(G) \leq K'(G)}$

All together now

$$\boxed{K(G) \leq K'(G) \leq \delta(G)}$$

$$\boxed{\kappa(G) \le \kappa'(G) \le \delta(G)}$$