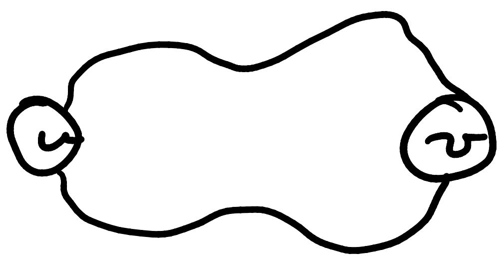
2-connectivity

→ must remove two vertices to disconnect G

Internally-disjoint paths

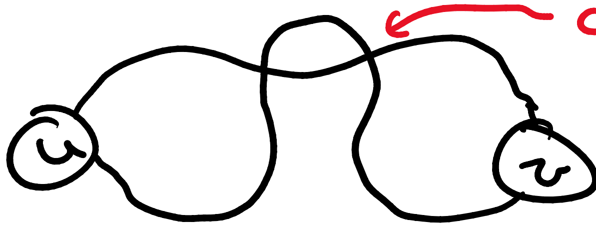
→ paths between some u,v that have no shared internal vertices



Internally edge-disjoint paths

→ paths between some u,v that have no shared internal edges

can have shared internal vertices



Whitney's Theorem (1932)

$|V(G)| \geq 3$

G is at least 2-connected

iff $\forall u,v \in V(G)$: $\exists$ u,v-idps iff $\forall u, v \in V(G): \exists u, v$-idps

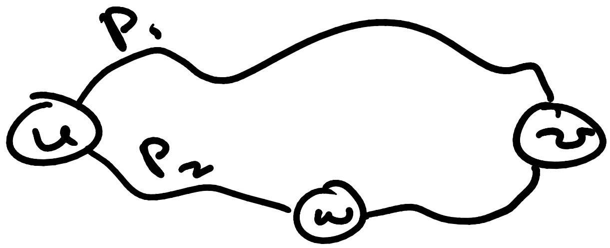
internally-disjoint paths

$\exists P_1 P_2$ idps $\forall u, v \in V(G)$

$\Rightarrow G$ is 2-connected

- consider any $u, v \in V(G)$
- consider some $w \in P_1$ or P_2

$\rightarrow$ removing w will ~~NOT~~ disconnect u from v



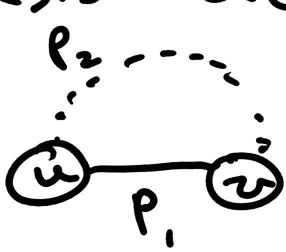
$\Rightarrow$ must remove 2 vertices to disconnect G

$\Rightarrow G$ is 2-connected ✓

G is 2-connected $\Rightarrow \exists P_1 P_2$ idps $\forall u, v \in V(G)$

Induction on distance $d(u, v)$

Basis: $d(u, v) = 1$



$P_1 = (u, v)$

$P_2 =$ any arbitrary path

$\rightarrow$ as $K'(G) \geq K(G)$,

deleting P will not

→ as ...

deleting P_1 will not disconnect u from v

Consider G with u, v s.t. $d(u,v) = n$ (2-connected)

→ ∃ at least one u,v-path P_1

- consider $w \in P_1$, $w \in N(v)$

$d(u,w) = n-1 = k$

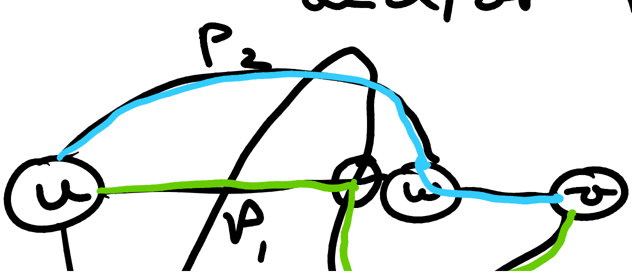
I.H. → ∃ P_1, P_2 idps from u to w

→ as G is 2-connected, ∃ P_3 from u to v (is it idps?)

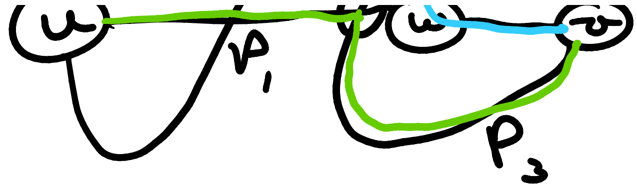
Case 1: P_3 does not intersect with P_1 or P_2

→ our two idps are P_3 and $P_1 + (w,v)$

Case 2: P_3 does intersect P_1 and/or P_2



- consider vertex x as last intersecting



intersecting vertex on P_3

- w.l.o.g. let's say $x \in P_1$

→ we can define 2 idps

$P_1' = P_2 + (w, v)$

$P_2' = P_1$ to x, P_3 from x to v ☑

QED

G is 2-connected $\Leftrightarrow \exists P_1 P_2$ idps $\forall u, v \in V(G)$

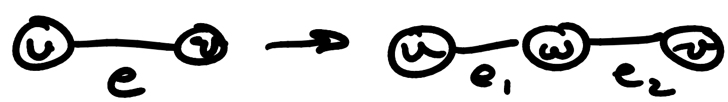
$\Leftrightarrow G$ is connected and has no cut vertex or cut edge

$\Leftrightarrow \forall u, v \in V(G): \exists C$ s.t. $u, v \in C$

$\Leftrightarrow \forall e, f \in E(G): \exists C$ s.t. $e, f \in C$

To prove the last statement, consider a subdivision of e and f

subdivision:



Note: subdivision does not impact 2-connectivity

→ $\exists$ 2 idps $w, \forall x \in V(G)$

$\to \exists$ 2 idps $w, \forall x \in V(G)$

So if we subdivide e and f

$\to$ we don't impact 2-connectivity

$\to$ if we have w from e and y from f

$\Rightarrow \exists C$ s.t. $w, y \in C$

$\Rightarrow$ that cycle equivalently would have e, f $\square$

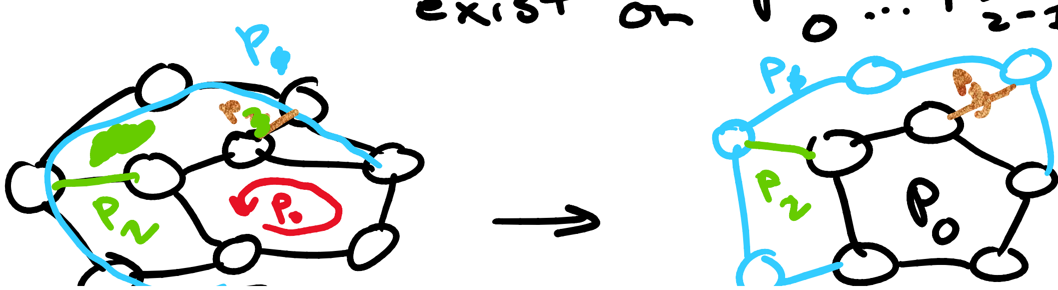
Ear decompositions

Open-ear decomposition is a decomposition on a 2-connected graph G s.t.

$O = P_0 P_1 P_2 \dots P_k$

P_0 = cycle

P_i = open path whose endpoints exist on $P_0 \dots P_{i-1}$

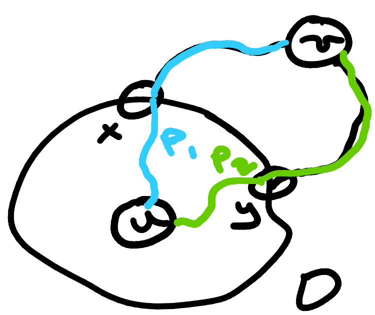


G is 2-connected $\iff$ G has a open-ear decomposition

G has D $\Rightarrow$ G is 2-connected

- first consider P_0, a cycle
  2-connected
- next consider some P_i
- consider some $v \in P_i$
- consider some $u \in P_0 \dots P_{i-1}$

$\rightarrow$ let's find our 2-idps



P_1 = v to x, then x to u along one of x,u-idps

P_2 = v to y, along one of y,u-idps

$\Rightarrow P_1$ and P_2 are internally disjoint

$\Rightarrow$ addition of an ear to a open ear decomposition does not

$\Rightarrow$ addition of an ear to a [illegible]
ear decomposition does not
affect 2-connectedness ✓

G is 2-connected $\Rightarrow \exists D$

- Let's build a decomposition

PROOF BY ALGORITHM

- Select any arbitrary $u, v \in V(G)$

  $\rightarrow \exists C$ s.t. $u, v \in C$

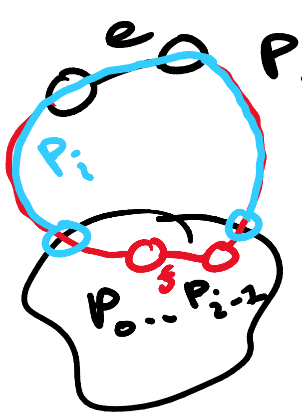
  $P_0 = C$

While $\exists e \in E(G) : e \notin P_0 \dots P_{i-1}$

  Consider any $f \in P_0 \dots P_{i-1}$

  $\exists C$ s.t. $e, f \in C$

  $P_i = e$ follows C until intersects with some edge in $P_0 \dots P_{i-1}$



$\Rightarrow$ this build D ☑

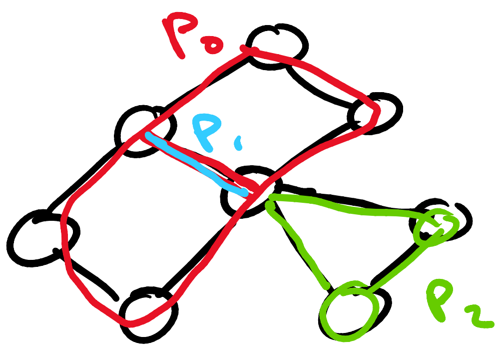
QED

Closed-ear decompositions

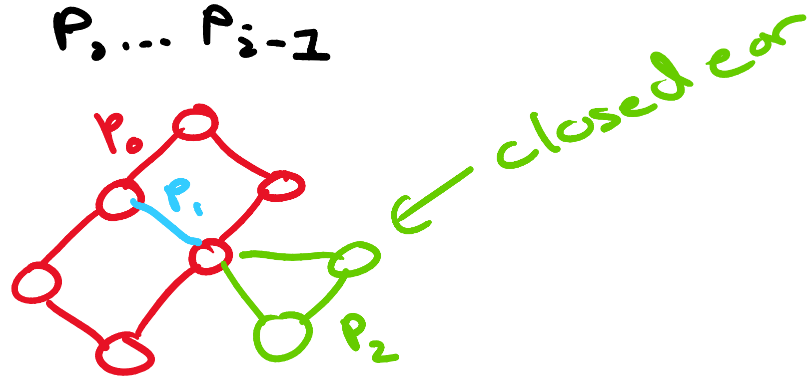
$D = P_0 \dots P_k$

P_0 = cycle

P_i = open OR closed path whose endpoints exist on $P_0 \dots P_{i-1}$



G



D

G is 2-edge-connected

proof is same as above

$\iff$ G has closed-ear decomp.

$\iff \forall u,v \in V(G) : \exists\ 2\ u,v$ internally edge-disjoint path

proof is same as whitneys

Biconnectivity

→ a biconnected graph has no cut vertex

★ → K_1 and K_2 are biconnected

Block decomposition of G

- blocks are maximal biconnected subgraphs of G
  - AKA biconnected components (BiCCs)
- articulation vertices are cut vertices of G that connect blocks
- bridges are cut edges of G

Using our blocks and articulation vertices, we can construct a block-cutpoint bipartite graph (forest)

Within a block-cutpoint graph:

$V(G_B)$ = blocks, articulation vertices

$E(G_B)$ = all articulation vertex

$E(G_B) =$ all articulation vertex membership in blocks