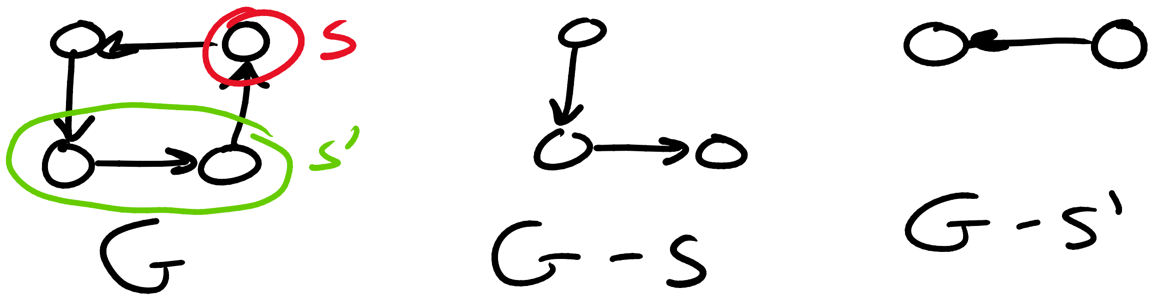
Digraph connectivity

vertex cut – a set $S \subseteq V(G)$ such that $G - S$ is not strongly connected



$K(G)$ = connectivity of digraph G

$= \min_{\forall S \subseteq V(G)} |S|$ aka minimum size of a vertex cut

kappa

If $k \leq K(G) \Rightarrow G$ is k-connected

Edge cut – a set $F \subseteq E(G)$ that separates $V(G)$ into two vertex sets $S, \bar{S}$

$\bar{S} = |V(G)| - S$

the size of this cuts is the number of edges from $S \to \bar{S}$

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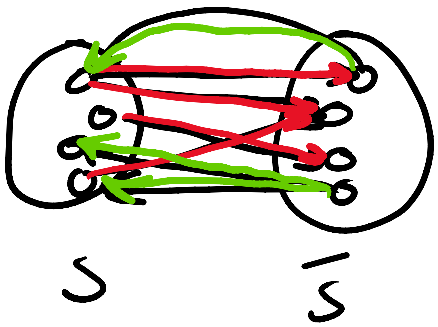
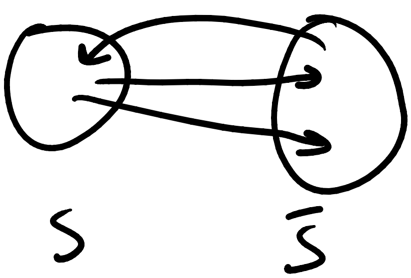


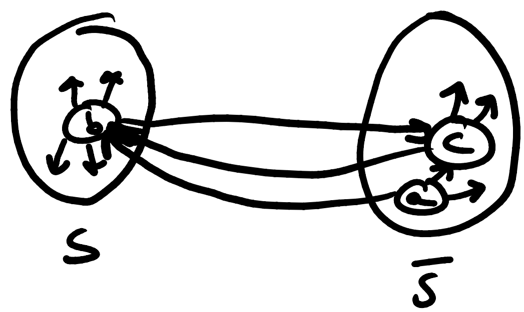
cut of $S \Rightarrow 4$
cut of $\bar{S} \Rightarrow 3$

$K'(G)$ = edge-connectivity of G

$$= \min_{S \subseteq V(G)} \text{cut of } S$$



cut = 2



$k = 2$ ✗
$k = 1$ ✓

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k-connectivity

x,y-separator – a set $S \subseteq V(G)$ s.t. $G - S$ has no x,y-path

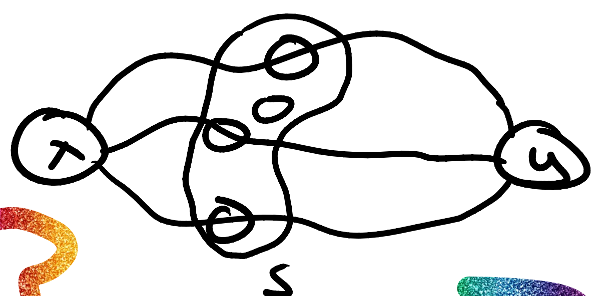
$K(G)$ = minimum x,y-separator over all $x, y \in V(G)$

$\kappa(x,y)$ = connectivity of x,y

= minimum size of an x,y-separator

$\lambda(x,y)$ = maximum number of internally-disjoint x,y-paths

First note: as every x,y-separator must contain a vertex from each idp $\Rightarrow \kappa(x,y) \geq \lambda(x,y)$



Big Question

does $\kappa(x,y) = \lambda(x,y)$?

Menger: it does: $\kappa(x,y) = \lambda(x,y)$ if $(x,y) \notin E(G)$

Let's use the power of strong induction to prove this

Induction on $|V(G)|$

Basis $P(2)$ x y

$\lambda(x,y)=0$

$\kappa(x,y)=0$

$\lambda(x,y)=\kappa(x,y)$ ✓

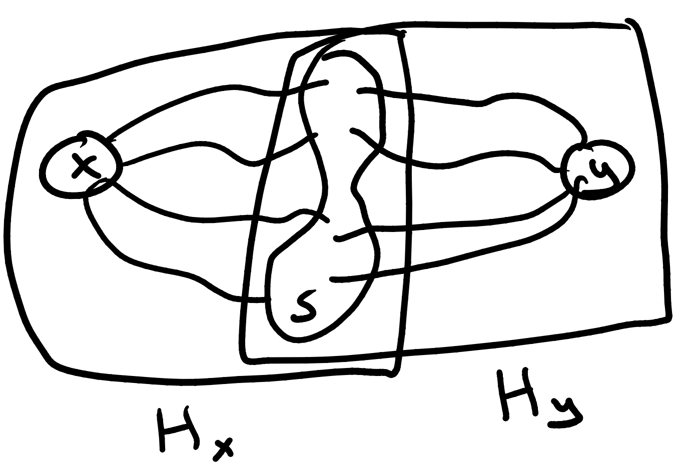
Assume we have G w/ $|V(G)|=n$

→ also assume we have $\kappa(x,y)=k=|S|$

=> construct k idps given our min separator S

Case 1: $S \neq N(y)$, $S \neq N(x)$

- consider X,S-paths
- consider Y,S-path



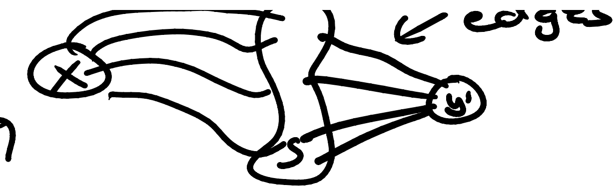
- define graph

$H_1 = H_x$ + vertex y' and edges from y' to all $s \in S$

paths

← edges

H_x $\qquad$ H_y



I.H. on H_1 gives us k idps

- Likewise, we can define $H_2 = H_y$ + vertex x' and edges from x' to all in S

→ I.H. on H_2 gives us k idps

Taken together ⟹ we have k x, S-idps and k S, y-idps, combined together we have our k x, y-idps

Case 2: $S = N(x)$ and/or $S = N(y)$

2a) $\exists v \notin \{x\} \cup \{y\} \cup \{N(x)\} \cup \{N(y)\}$

- consider $G - v$
→ I.H. on $G - v$

Note: v is not on a min. cut

⟹ k-idps on $G - v$

⟹ k-idps on G

$\Rightarrow$ k-idps on G ...

$\Rightarrow$ k-idps on G

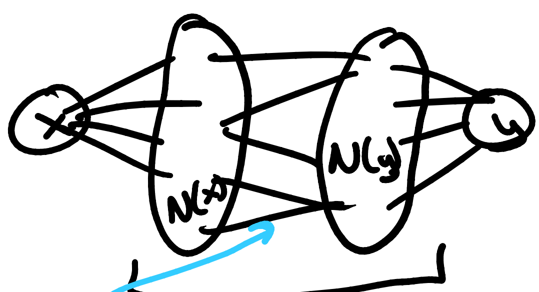
2b) if $\exists v \in N(x) \cap N(y)$

- Consider $G - v$

$\rightarrow$ I.H. on $G - v \Rightarrow (k-1)$-idps

$\Rightarrow$ to get back to G, we can add path (x, v, y) to get k-idps on G

2c) Otherwise, both $N(x)$ and $N(y)$ are x,y-separators

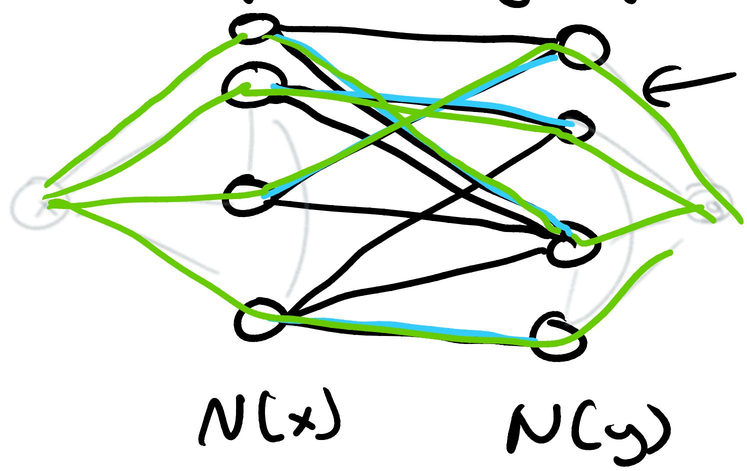


$\leftarrow$ only possible configuration after 2a, 2b

we have a bipartite graph if we consider only edges between some $a \in N(x)$ and $b \in N(y)$

Note! every x,y-path uses an edge in this bipartite graph

How can we guarantee that there will be k-idps using k edges in our bipartite graph?



Meganote: this is essentially reduced to a matching problem

Ultranote: as each of N(x) and N(y) are separators, they are also vertex covers in our bipartite graph

↳ minimum covers is bounded below by the minimum of $|N(x)|$ or $|N(y)| = |S|$

$\Rightarrow$ from König-Egerváry

min cover = max match

$\Rightarrow$ we have a maximum match of size k

$\Rightarrow$ we can construct our k-idps on G by taking

$\{x \to N(x)\}$ $\{$max match$\}$ $\{N(y) \to y\}$

$\Rightarrow$ we have $|S| =$ k-idps

QEWL

Same but different for edge-connectivity

$\kappa'(x,y) =$ x,y-edg-connectivity

$=$ minimum x,y edge cut

$\lambda'(x,y) =$ maximum number of edge-disjoint path

$$\Rightarrow K'(x,y) = \lambda'(x,y)$$

G is k-edg-connected if

$\forall x,y \in V(G): K'(x,y) \geq k$

$\lambda'(x,y) \geq k$