Lecture 17 - Epidemiology

Thursday, March 24, 2022 9:58 AM

Warning: MATH

Epidemiology: what is this stuff?

→ study of disease patterns in a population

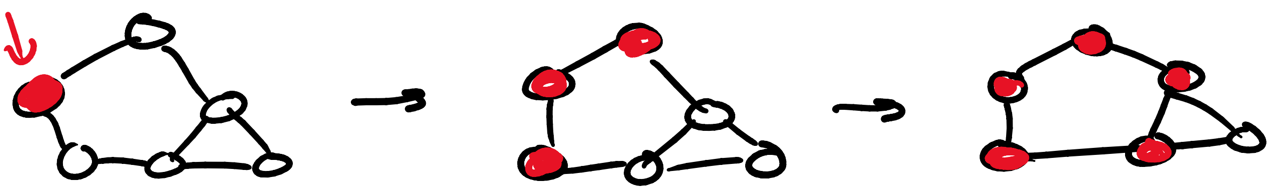
→ dynamics of spread, etc.

How is this relevant to graph theory?

→ we're considering models defined implicitly on a random graph

→ simulations can be run on a explicit graph

infectious



(diffusive process)

How do these graphs look?

- Models → homogeneous (Erdös-Rényi)
  → heterogeneous (Chung-Lu)
- Simulations (agent-based)
  → arbitrarily complex

- simulations (agent-based)

→ arbitrarily complex

Math O'clock

↳ mathematical models for the spread of a disease

Classic model: compartmental model
- population is separated into "compartments" based on some state
- spread is captured via changes in sizes of each compartment

Classic of the classics aka O.G. model boi

→ SIR

S: susceptible, can be infected

I: infectious, can spread disease

R: removed, immune and non-contagious

other variations → SIS, SEIR

Model dynamics of SIR

→ how does our subpopulations change from $S \to I \to R$

$S \to I$

$$\frac{dS}{dt} = \text{change in } S \text{ over time}$$

$$\frac{dI}{dt} = \text{'' } I \text{ ''}$$

$$\frac{dR}{dt} = \text{'' } R \text{ ''}$$

Parameters affecting the model

- Population N (often divided out)
- contact rate → assume an Erdös-Rényi homogeneous network
- probability of transmission

$$\to \beta = \frac{\text{contacts}}{\text{time}} * \frac{\text{prob. transmission}}{\text{contact}}$$

$$\beta = \frac{\text{prob. transmission}}{\text{time}}$$

time

→ in a given time, effectively its the number of trasmissions

- duration of infection $= T$

→ $\gamma = \frac{1}{T}$ → rate of recovery

Let's get definin'

$$\frac{dS}{dt} = -\beta \frac{IS}{N}$$ ← possible I↔S interactions

↑ transmission rat

$$\frac{dI}{dt} = \beta\frac{IS}{N} - \gamma I$$

← rate of recovery

→ # of infectuous who recover

$$\frac{dR}{dt} = \gamma I$$

Note: $$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

$S + I + R = N$ for all t

→ Our dynamics only depend on β, γ

↳ R = basic reproductive number

$\hookrightarrow R_0$ = basic reproductive number
= # of new infections from a single infection

$$R_0 = \beta T = \frac{\beta}{\gamma}$$

$R = R_{eff}$ = reproductive number at some time t

obviously: $R \leq R_0$

why: $S(t)$ goes down over time

Model simplifications and limitations

- N = fixed, births & deaths ignored (vital dynamics)
- ignore reinfectibility (SIS)
- homogenous mixing
  reality → contact patterns are skewed (superspreaders)
- assumes static behavior

Our model as a system

Generally: we can get rid of N

lil' s

Generally we can get:

lil' s

$$\frac{ds}{dt} = -\beta i s \qquad s(0) \geq 0$$

$$\frac{di}{dt} = \beta i s - \gamma i \qquad i(0) \geq 0$$

IVP

$$r(t) = 1 - s(t) - i(t)$$

Let's consider the behavior of this system as $t \to \infty$

What can that tell us?

total infected: $s(0) - s(\infty)$

peak infected: $\max_t i(t)$

peak over all $t = 0 \dots \infty$

We can get some "nice" solutions in terms of $i(0), i(\infty), s(0), s(\infty)$

consider $\frac{di}{dt} \Big/ \frac{ds}{dt}$

$$\Rightarrow \frac{di}{ds} = -1 + \frac{\gamma}{\beta s} = \underbrace{-1 + \frac{1}{R_0 s}}_{\boxed{\text{MATH}}}$$

$\boxed{\text{MATH}}$

$$i(\infty) - i(0) = -(s(\infty) - s(0)) + \frac{\ln\left(\frac{s(\infty)}{s(0)}\right)}{R_0}$$

we can assume $i(\infty) = 0$

$i(0) \approx 0$

$s(0) \approx 1$

$$\boxed{0 = -s(\infty) + 1 + \frac{\ln(s(\infty))}{R_0}}$$

from this, we can determine $s(0) - s(\infty)$ aka total infected

How: get $s(\infty)$ by finding the roots

consider $R_0 = 2$

$\rightarrow s(\infty) = 0.2 \rightarrow$ total infected $= 0.8$

Note: total infected is solely a function of R_0

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what about $\max_t i(t)$?

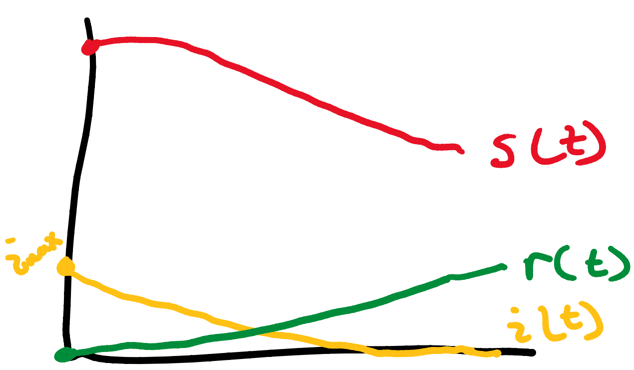
Depends on R_0

$R_0 = 1 \rightarrow$ static $i(t)$ so $i_{max} \approx i(0)$

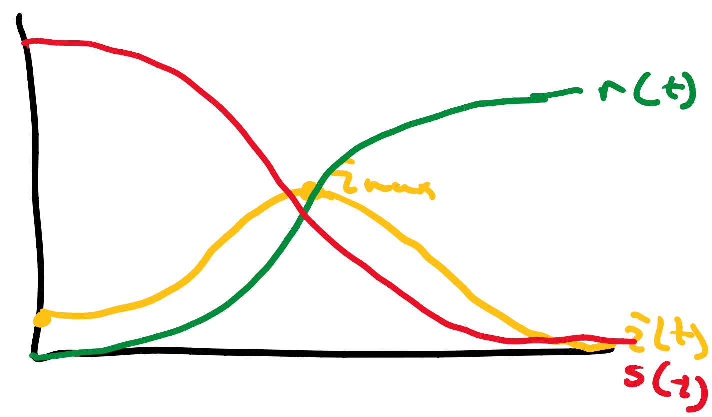
$R_0 < 1 \rightarrow i(t)$ decays so $i_{max} = i(0)$

$R_0 > 1 \rightarrow i(t)$ grows then decays

$R_0 < 1$



$R_0 > 1$



What is that i_{max}?

$$\boxed{\text{MATH}} \rightarrow i_{max} = i(0) + s(0) - \frac{1}{R_0} - \frac{\ln(R_0 s(0))}{R_0}$$

assuming $i(0) \approx 0$

$s(0) \approx 1$

$$\boxed{i_{max} = 1 - \frac{1}{R_0} - \frac{\ln(R_0)}{R_0}}$$

if $R_0 = 2 \rightarrow i_{max} = 0.15$

Note that i_{max} is also a

→ Note that i_{max} is also a function of R_0

Q: how can we determine or estimate R_0?

Imagine at the conclusion of some epidemic

- we can measure $s(\infty)$ or $r(\infty)$
- we can estimate $s(0)$ or $r(0)$

$$\boxed{\text{MATH}}$$

assuming
$s(0)=1$
$i(0)=0$
$i(\infty)=0$

$$s(\infty)-1=\frac{\ln(s(\infty))}{R_0}$$

$$R_0=\frac{\ln(s(\infty))}{s(\infty)-1}$$

In reality, $s(0)<1$

$$s(\infty)-s(0)=\frac{\ln\left(\frac{s(\infty)}{s(0)}\right)}{R_0}$$

$$\boxed{\text{MATH}}$$

MATH

$$\boxed{R_0 = \frac{\ln\left(\frac{s(0)}{s(\infty)}\right)}{s(0) - s(\infty)}}$$

Note: this is just an estimate, assuming we can measure $s(0)$, $s(\infty)$

What about non-homogeneous aka Chung-Lu networks?

Really: is consider the pairwise interaction between all degrees

Recall: Chung-Lu graph are really just layered $G(n,p)$ Erdös_Rényi graphs for each possible i,j degree pair

important: $p_{i,j} = \frac{ij}{2m}$

We can reformulate our SIR as a sum over all degree pairs

← back 2 bigS

← degree k

attach. prob.

$$\frac{dS_k}{dt} = -\beta \underbrace{k S_k(t)}_{} \sum_l p_{k,l} \frac{I_l(t)}{N}$$

← pop. w/

$$\frac{dS_k}{dt} = -\beta \underbrace{k S_k(t)}_{\text{total contacts}} \sum_l \underbrace{p_{k,l}\frac{I_l(t)}{N_l}}_{\text{prob. of contact to an infectious } l}$$

β ← same transmission rate

N_l ← pop. w/ degree l

$$\frac{dI_k}{dt} = \beta k S_k(t) \sum_l p_{k,l} \frac{I_l(t)}{N_l} - \gamma I_k(t)$$

$$\frac{dR_k}{dt} = \gamma I_k(t)$$

Another way to estmate R_0

$$R_0 \approx \frac{I_{n+1}}{I_n} \approx \frac{I_2}{I_1}$$ with a single vertex of degree k infected with prob $\frac{N_k}{N}$

$$I_{l,1} = \bar{p} \sum_k p_{l,k} \frac{N_k}{N}$$

$\bar{p}$ ← transmission prob. per edge

$I_{l,1}$ ← num in l infected after 1st generation

$$= \bar{p} \frac{l}{2M} \underbrace{\sum_k k \frac{N_k}{N}}_{\text{avg degree} = \langle k \rangle}$$

also note:

$\langle k \rangle = \frac{2M}{N}$

$2M = \langle k \rangle N$

$$= \frac{\bar{p}\, l}{\cancel{\langle k \rangle} N} \cancel{\langle k \rangle}$$

$$I_{\ell,1} = \frac{\bar{p}\,\ell}{N}$$

For second generation

$$I_{m,2} = \bar{p} \sum_{\ell} p_{m,\ell} I_{\ell,1}$$

So we can take $\frac{I_2}{I_1}$

summed over all degree classes

$\downarrow$

$\boxed{\text{MATH}}$ "easy to show"

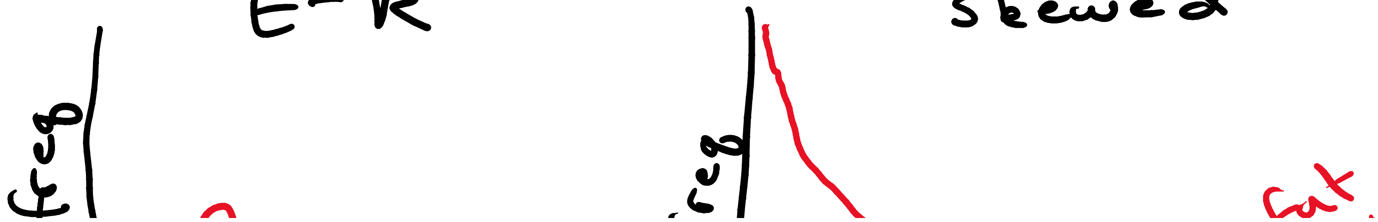
$$\frac{I_2}{I_1} \Rightarrow R_0 = \bar{p}\,\frac{\langle k^2 \rangle}{\langle k \rangle}$$ $\leftarrow$ second moment of degree distribution

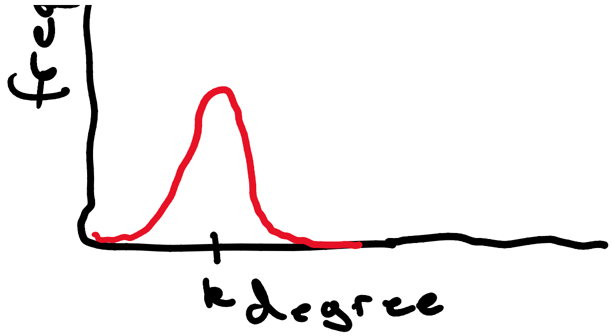
consider an Erdös-Rényi

$\langle k^2 \rangle \approx \langle k \rangle^2$ so $\underbrace{R_0 = \bar{p}\langle k \rangle}$

What about a skewed distribution?

E-R skewed





For skewed distributions

$$\langle k^2 \rangle \gg \langle k \rangle^2$$

due to "fat tail" of real, skewed, distributions

$\Rightarrow$ R_0 assuming E-R < R_0 assuming real distribution

Note: fat-tailed sheep