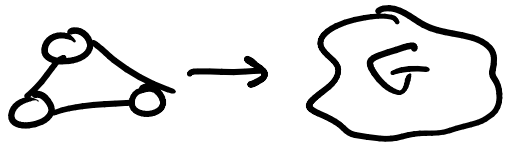
Lecture 18 - Graph Coloring

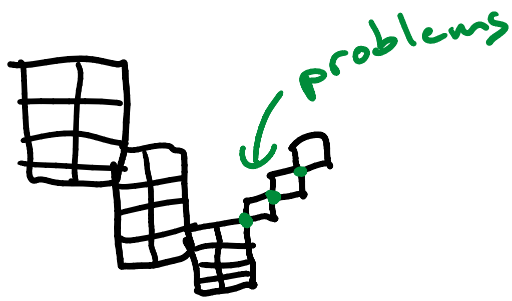
Monday, March 28, 2022 9:54 AM

Applications of this stuff

1. (sub)graph isomorphism



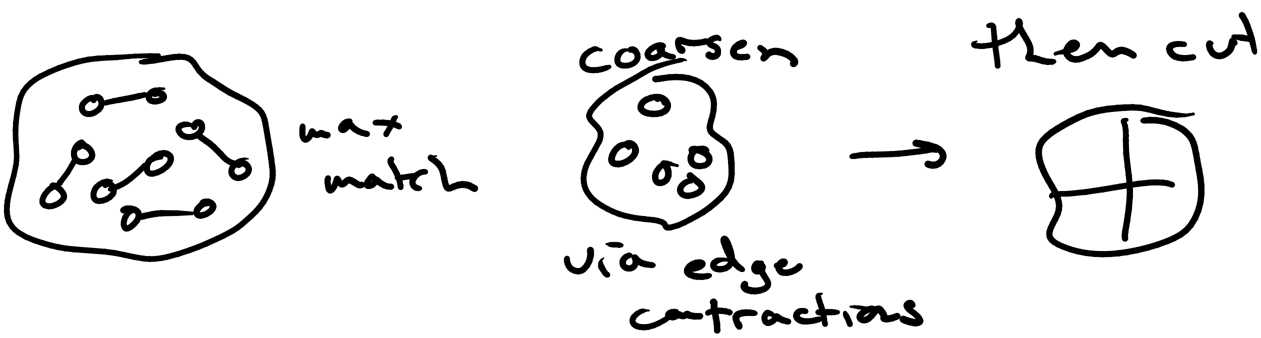
2. biconnectivity



ice sheet

3. Matching

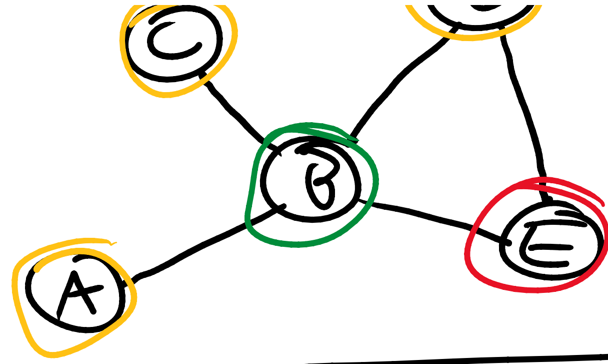
→ Graph partitioning



4. Coloring

dependency

dependency graph



Graph coloring

k-coloring of G is a labeling $f: V(G) \to S$, $k = |S|$

proper coloring: a k-coloring of G s.t. no neighboring vertices have the same color

G is k-colorable if it can be properly colored with k colors

Chromatic number of G $\to \chi(G)$

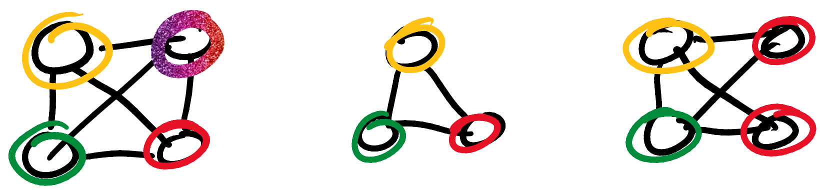
$\chi(G)$ = the minimum k for which G is k-colorable

Optimal coloring of G is a k-coloring for $\chi(G) = k$

G is <u>color-critical</u> if for all subgraphs $H \subseteq G$, $H \neq G$

$$\chi(H) < \chi(G)$$

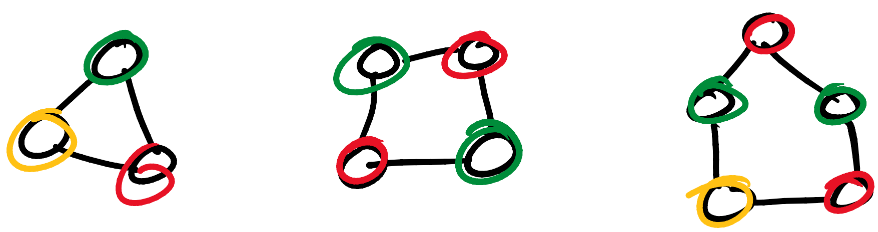
Note: all cliques are color-critical



K_4 $\quad$ $K_4 - v$ $\quad$ $K_4 - e$

$$\chi(K_4) = 4 \rightarrow 3 \longrightarrow 3$$

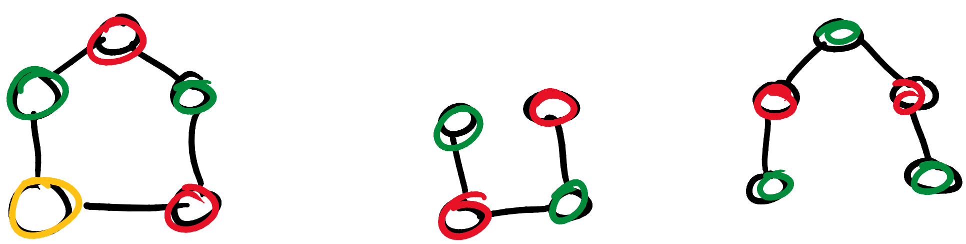
Note 2: odd cycles are color-critical



$$\chi(C_n : n = \text{odd}) = 3$$

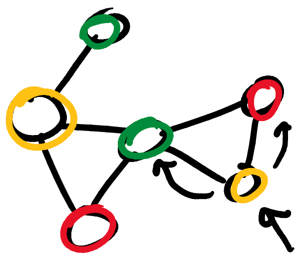
$$\chi(C_n : n = \text{even}) = 2$$

color-criticality of odd cycles:



Greedy coloring algorithm

all vertices have empty color
for all vertices in some order:
    color vertex with "least"
    color that doesn't exist
    in its neighborhood



G

Let's talk bounds
(on chromatic numbers)

In a generic non-null graph G:

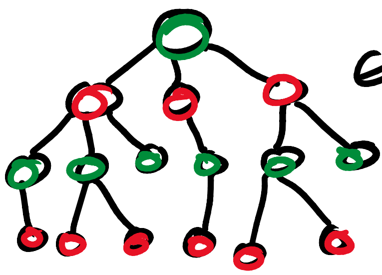
$$1 \le \chi(G) \le |V(G)|$$

If G is non-empty: (has edges)

$$2 \le \chi(G)$$

If G is a tree

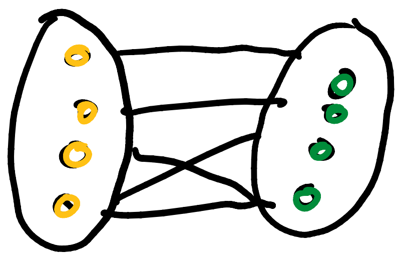
 need 2 colors,



← only need 2 colors, regardless of tree structure

$\chi(T) = 2$ for tree T

If G is bipartite



$\chi(B) = 2$ for bipartite B

If G is a clique K_n

$\chi(K_n) = n$

If $\alpha(G)$ = the size of the largest independent set of G aka "independence number"

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

If $\omega(G)$ = size of the largest clique in G, aka "clique number"

$$\chi(G) \geq \omega(G)$$

Considering $\Delta(G)$ and our greedy coloring algorithm greedy coloring algorithm

$$\chi(G) \le \Delta(G) + 1$$

Can we improve on this upper bound?

Brooks says: Yes

Brooks: $\chi(G) \le \Delta(G)$

except for odd cycles and cliques

To prove: construct an ordering for greedy coloring s.t. we can guarantee each vertex we color has at most $k-1 = \Delta(G)-1$ prior colored neighbors

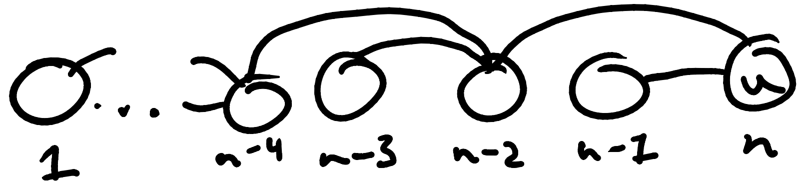
Case 1: G is not k-regular

($\forall v \in V(G): d(v) = k$)

- consider $u \in V(G): d(u) < \Delta(G)$
- grow a spanning tree from u
- apply order in reverse

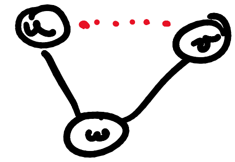
→ every vertex is guaranteed at least on higher-ordered vertex

→ order →



Case 2: G is k-regular

- consider $(u, v) \in N(w)$ s.t. $(u, v) \notin E(G)$

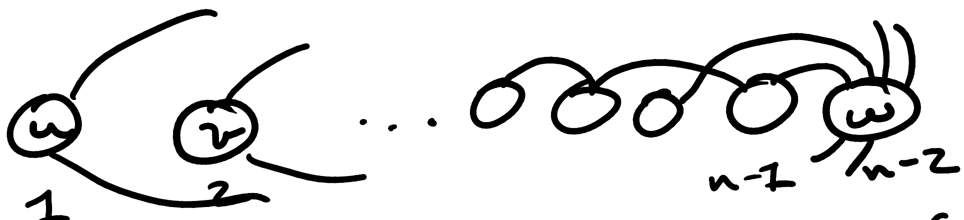


→ can we guarantee that there exists two non-adjacent vertices?

Yes → G is not a clique

To construct our order
- u, v are listed first
- w is listed last
- we grow our spanning tree from w

→ order →



To color → fix $C(u) = C(v)$ ← color of u equals color of v

and then process greedy coloring

From this, at most $\Delta(G) - 1$ prior neighbors → max color of $\Delta(G)$

neighbors → max color of $\Delta(G)$

How tight are those bounds?

→ Not very

Note: a tree can have an arbitrarily large max degree

→ but $\chi(T) = 2$ for all trees T

$$\chi(T) = 2 \lll\lll\lll \Delta(T)$$

what about lower bounds?

$2 \leq \chi(G)$ if G is non-empty

$\omega(G) \leq \chi(G)$ in general

First: consider some graph where we can guarantee inequality in the above

$$\chi(G) \geq n+3$$

↳ this tells us the $\omega(G) \leq \chi(G)$ bound can be loose

How loose can this be?

- Consider some triangle-free graph G

$$\omega(G) = 2 \leq \chi(G)$$

→ and how large a triangle-free graph can be

⟹ we can come up with an iterative construction that keeps a graph triangle-free and increasing its chromatic number

aka Mycielski's Construction

→ Given triangle-free graph G, with $\chi(G) = k$, we can construct triangle-free G' with $\chi(G') = k+1$

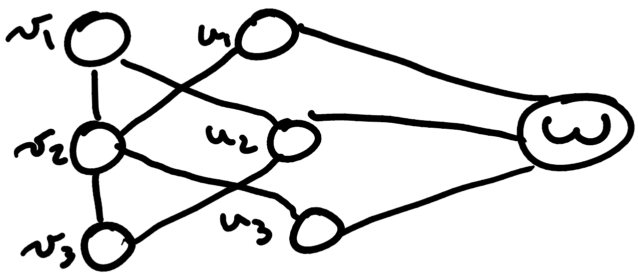
consider $v_1 v_2 \ldots v_n \in V(G)$

create $u_1 u_2 \ldots u_n$

add edges between u_i and all $v_j \in N(v_i)$

create w

add edges from w to all u_i



note: we don't create any triangles

note 2: a coloring of u_i requirs same number of colors as for v_i

note 3: we require an additional new color for w

$\Rightarrow$ we keep $\omega(G) = 2$ while increasing our chromatic number by 1

$\Rightarrow \omega(G) = 2 <<<<<<<<<<<<<<<<<< \chi(G)$

Takeaway: our bounds can tell us very little in the general case ¯_(ツ)_/¯

↳ why?

↳ why?