Lecture 20 - Counting colors

Monday, April 4, 2022 10:01 AM

$\chi(G,k)$ = # of ways to color graph G with k colors

obviously

$\chi(G,k) = 0$ if $k < \chi(G)$

Consider clique K_n and some k (really $\chi(K_n,k)$)

First color any vertex with one of k possible colors

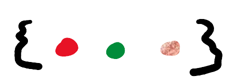
color second vertex with any of k-1 possible color

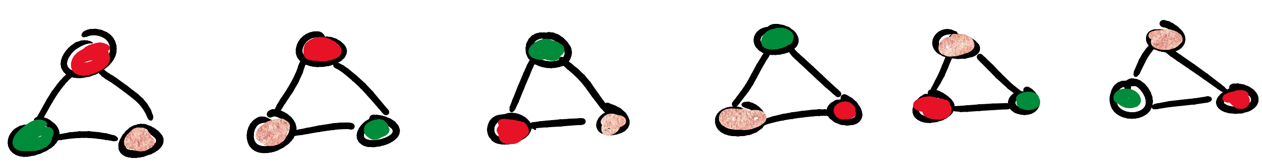
...

color final vertex with k-n+1 possible colors

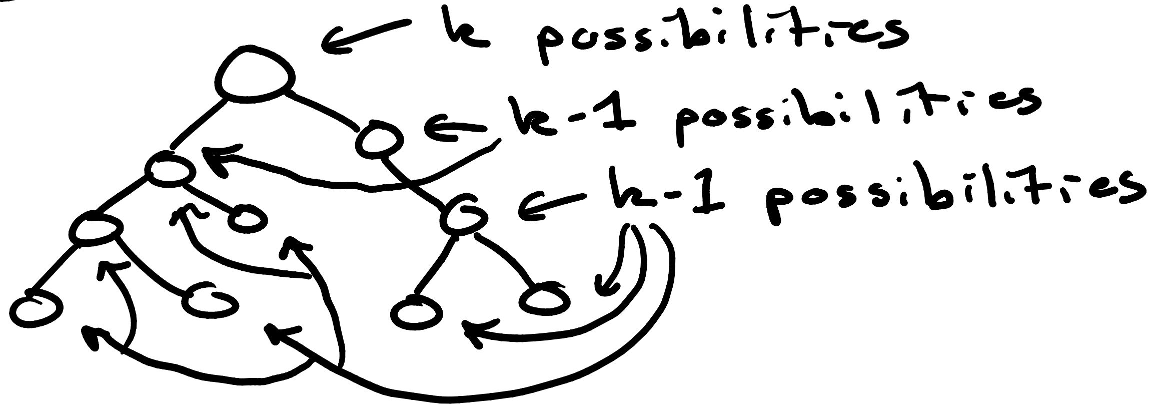
$\chi(K_n,k) = k(k-1)(k-2)\dots(k-n+1)$

consider K_3 and $k=3$





Let's talk trees



$$\chi(T,k) = k(k-1)(k-1)\dots(k-1)$$

$$= k(k-1)^{n-1}$$

$\chi(T,k)$ = chromatic polynomial

General form

$$\chi(G,k) = \sum_{r=1}^{n} p_r(G)\, k_r$$

$p_r(G)$ = # of ways to partition G into r independent sets (quite tough to calculate)

k_r = # of ways to assign colors to these independent set

$$= k(k-1)(k-2)\dots(k-r+1)$$

Consider C_4 and its chromatic polynomial

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$p_1(C_4) = 0 \quad p_2(C_4) = 1 \quad p_3(C_4) = 2 \quad p_4 = 1$

$$\chi(C_4, k) = \sum_{r=1}^{n} p_r(G)\, k_r$$
$$= 0 + 1\,k(k-1)$$
$$+ 2\,k(k-1)(k-2)$$
$$+ 1\,k(k-1)(k-2)(k-3)$$

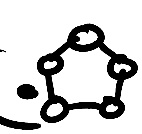
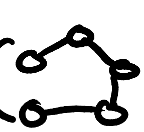
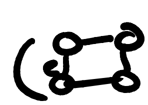
Fundamental Reduction Theorem

$$\chi(G, k) = \chi(G-e, k) - \chi(G\cdot e, k)$$
$$e = (u, v) \in E(G)$$

$\chi(G-e, k) =$ # of ways to color G with $C(u) = C(v)$ and $C(u) \neq C(v)$

$\chi(G\cdot e, k) =$ # of ways to color G with $C(u) = C(v)$

Consider the above with C_5

$\chi($  $, k) = \chi($  $, k) - \chi($  $, k)$

tree

tree clique

$$= k(k-1)^4 - (\chi(\cdot, k) - \chi(\cdot, k))$$

$$= k(k-1)^4 - (k(k-1)^3 - k(k-1)(k-2))$$

$$= k(k-1)^4 - k(k-1)^3 + k(k-1)(k-2)$$

$$\chi(C_5, 1) = 0$$

$$\chi(C_5, 2) = 2(1)^4 - 2(1)^3 + 0 = 0$$

$$\chi(C_5, 3) = 3(2)^4 - 3(2)^3 + 3(2)(1)$$

$$= 48 - 24 + 6$$

$$= 30 \checkmark$$

thus, $\chi(C_5) = 3$

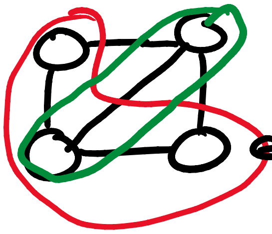
Simplicial vertices

A <u>simplicial vertex</u> is a vertex v

where $N(v) \cong K_n$

aka $N(v)$ is a clique

 ← not simplicial



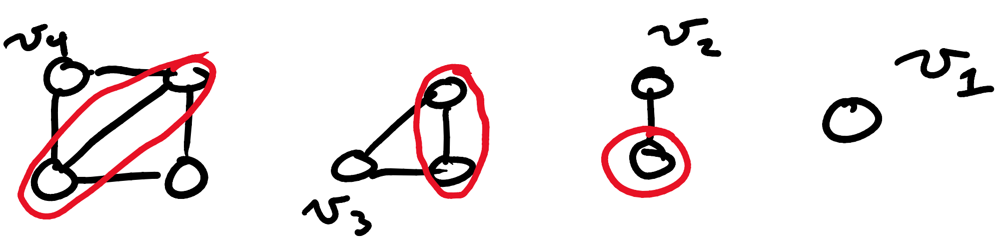
← not simplicial

← is simplicial

Note: v forms a larger clique with $N(v)$

<u>Simplicial elimination ordering</u> (SEO)

an ordering $\{v_n, v_{n-1}, \ldots, v_1\}$ of all $v \in V(G)$ for deletion such that each v_i is simplicial in the remaining graph induced on $\{v_i, v_{i-1}, \ldots, v_1\}$



Note: we can construct a chromatic polynomial using this ordering

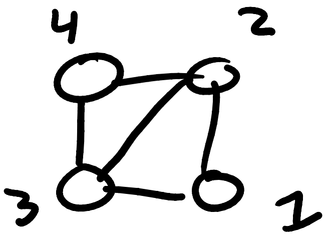
To get $\chi(G,k)$ using a SEO

→ add $v_1, v_2 \ldots v_n$ to $G_i : G[\{v_1 \ldots v_i\}]$

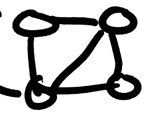
$$\chi(G,k) = \prod^{n} (k - d^{i}(v_i))$$

$$\chi(G,k) = \prod_{i=1}^{n} \left(k - d'(v_i)\right)$$

degree of v_i in G_i



$$\left.\begin{aligned} d'(v_1) &= 0 \\ d'(v_2) &= 1 \\ d'(v_3) &= 2 \\ d'(v_4) &= 2 \end{aligned}\right\} \to k(k-1)(k-2)^2$$

$\chi($  $, k) = k(k-1)(k-2)^2$