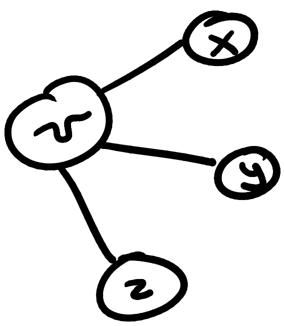
Question: $\forall v \in V(G)$, is v in at most 2 subgraphs in our decomposition?

- consider $v \in S_i\ S_j\ S_k$

$\{x, y, z\} \in N(v)$

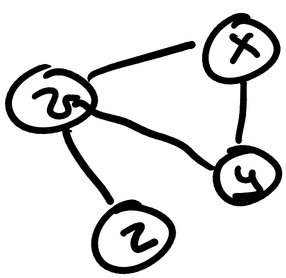
$x \in S_i,\ y \in S_j,\ z \in S_k$

Case 1: no edges $(x,y), (y,z), (x,z)$



→ a claw, so we can't have this configuration

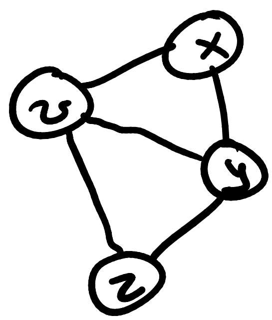
Case 2: edge (x,y) exists



→ an odd triangle, but our decomposition specified odd triangle are in a

specified such
triangle are in a
single S_e

Case 3: edges (x,y) and (y,z) exist



→ we have two even triangles, regardless of choice of v

Case 4: edges (x,y), (y,z), (z,x) exist

→ we have K_4, which would only be in one of S_e

⟹ taken together, along with our assumed decomposition, v can be in at most two subgraphs in that decomposition

$\Rightarrow \exists H$ s.t. $G = L(H)$ $\square$

$\Rightarrow \exists H$ s.t. $G = L(H)$ $\square$

Characterization of G:

$\begin{cases} \text{G has no double odd triangles} \\ \text{G has no claws} \end{cases}$

→ Forbidden subgraphs

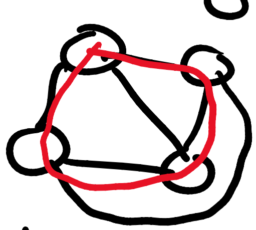
Hamiltonian Cycles

Hamiltonian Graph: graph that contains a Hamiltonian cycle

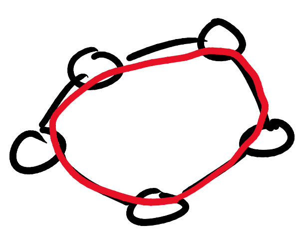
Hamiltonian Cycle: spanning cycle

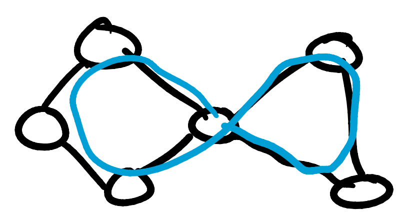
Hamiltonian Path: spanning path

What graphs are Hamiltonian:



cliques

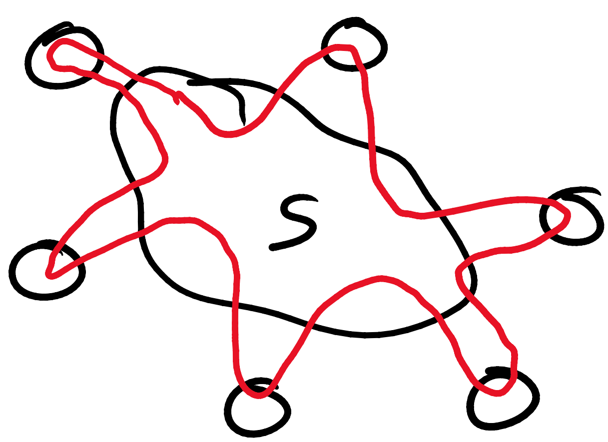




cliques $K_{n \geq 3}$ C_n Eulerian?

Necessary conditions for G

- 2-connected $\rightarrow$ cycle can't pass through a cut vertex
- connected (obvious)
- If G is bipartite, then $|X| = |Y| \rightarrow$ a cycle has to hop between sets an equal number of time
- If $c(G)$ is # components of G, the $c(G-S) \leq |S| \; \forall S \subseteq V(G)$



These condition are necessary.

Q: what about sufficient condition????

Sufficient conditions for Hamiltonian graphs

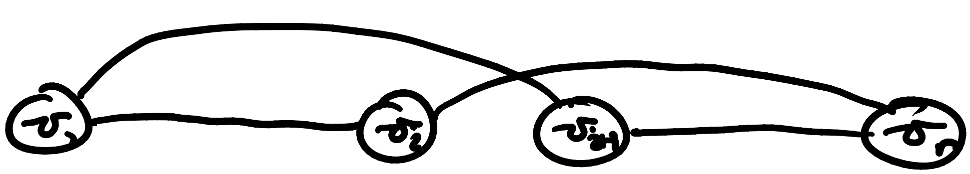
if $|V(G)| \geq 3$ and $\delta(G) \geq \frac{|V(G)|}{2}$

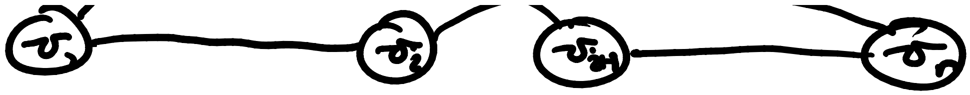
- consider maximum non-Hamiltonian G'
  - $\rightarrow G' + e$ = Hamiltonian
  - $\rightarrow G'$ has a Hamiltonian path

- consider this path in some order $v_1 v_2 \dots v_n$

If along this path $\exists v_i v_{i+1}$ s.t. $v_i \in N(v_n)$, $v_{i+1} \in N(v_1)$

$\rightarrow$ we can create a cycle





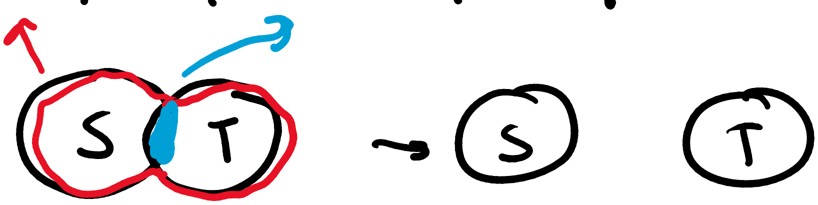
- define $S = \{ i : (v_1, v_{i+1}) \}$

$T = \{ i : (v_n, v_i) \}$

show: $|S \cap T| \geq 1$

$\Rightarrow$ we have a cycle

$|S \cup T| + |S \cap T| = |S| + |T|$



$|S| + |T| = d(v_1) + d(v_n) \geq |V(G)|$

$|S \cup T| + |S \cap T| \geq |V(G)|$

$|S \cup T| < |V(G)|$ b.c. we assume no (v_1, v_n) edge

$|S \cap T| \geq 1$

$\Rightarrow$ we have a spanning cycle $\square$

If $\forall u,v \in V(G)$ $(u,v) \notin E(G)$
and $d(u) + d(v) \geq |V(G)|$

G is Hamiltonian iff
$G + (u,v)$ is Hamiltonian

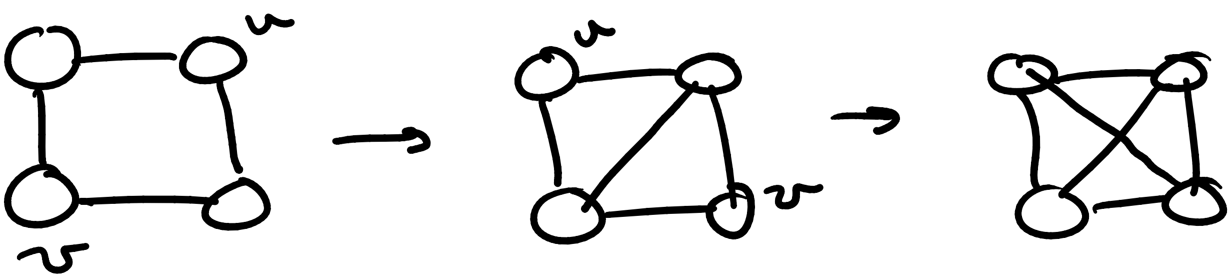
($\Rightarrow$) trivial, as adding an edge won't break a spanning cycle

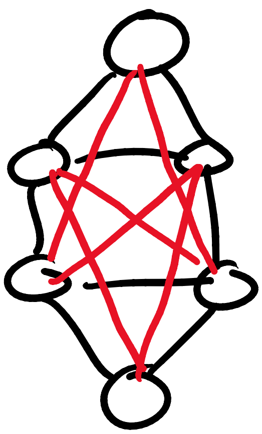
($\Leftarrow$) this follows from our prior proof since $|N(u) \cap N(v)| \geq 1$

We can use the above to determine the <u>closure</u> of G

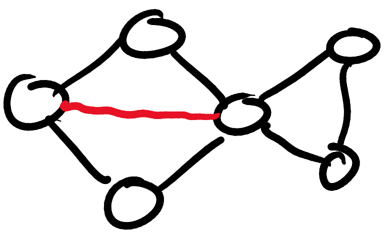
If G's closure is Hamiltonian
$\Rightarrow$ G is Hamiltonian

Closure of G:
add (u,v) $\forall u,v \in V(G)$
s.t. $d(u) + d(v) \geq |V(G)|$





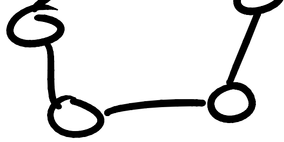
$\to K_6 \to$ Hamiltonian



closure of fish graph <u>NOT</u> Hamiltonian

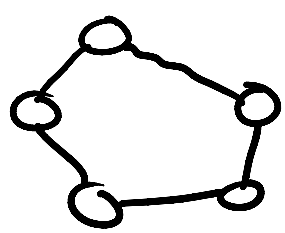
Sufficient condition:
If G's closure is a clique, then G is Hamiltonian

Note: above is not necessary



C_5

$\to$



Closure of C_5

C_5 Closure of C_5

Q: is the closure of G well-defined?

Consider

$e_1 e_2 \dots e_i$ and $f_1 f_2 \dots f_j$ are edges added to create the closures of $G \to G_e, G_f$

→ since e_1 can be added for G_e, it must also be added for G_f as some f_k

→ If any e_l depends on e_1, there is equivalently some f_m that depends on f_k and <u>will</u> be added to G_f

=> all the same edge will eventually be added eventually be added to $G_e, G_f \rightarrow G_e \cong G_f$ $\square$

We can define a numerical relation on the degree sequence of G s.t. we know G's closure is a clique and therefore G is Hamiltonian

$\Rightarrow$ Chvátal's Condition

consider G with degrees

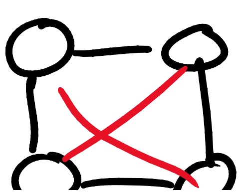
$$d_1 \le d_2 \le \dots \le d_n$$

if $i < \frac{n}{2}$ implies $d_i > i$

or $d_{n-i} \ge n-i$

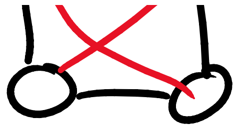
$\Rightarrow$ closure of G is K_n

$\Rightarrow$ G is Hamiltonian

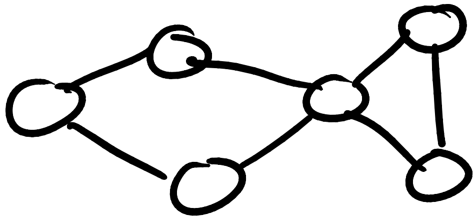


$S = 2, 2, 2, 2$ $\quad i = 1$

$i = 1, 2, 3, 4$ $\quad d_1 = 2 > 1$



$i = 1, 2, 3, 4$ $\quad d_1 = 2 > 1$

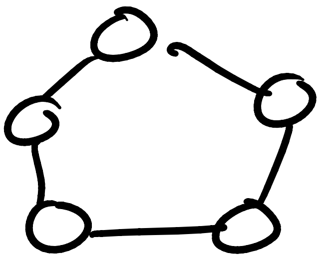


$S = 2, 2, 2, 2, 4$

$i = 1, 2$

$i = 1$ $\quad i = 2$

$d_1 = 2 > 1$ $\quad d_2 = 2 \not> 2$



$S = 2, 2, 2, 2, 2$

$i = 1, 2$

$i = 1$ $\quad i = 2$

$d_1 = 2 > 1$ $\quad d_2 = 2 \not> 2$

$d_3 = 2 \not\geq 3$

d_{n-i} $\quad n-i$

Chvatal's condition is <u>sufficient</u> but not <u>necessary</u>

Hamiltonian path $\rightarrow$ spanning path

<u>Graph join</u> between G and H, notationally as $G \vee H$, is adding an edge (u,v)

$$\forall u \in V(G)$$
$$\forall v \in V(H)$$

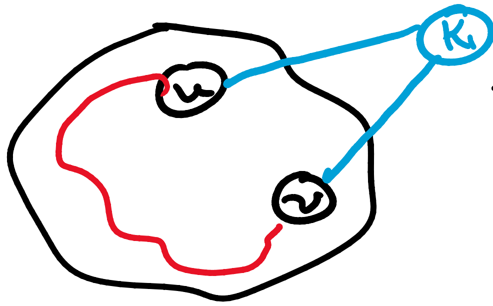
If $I = G \vee H$

$$V(I) = V(G) \cup V(H)$$
$$E(I) = E(G) \cup E(H) \cup \{\text{all } e = (u,v)\ \forall u \in V(G),\ \forall v \in V(H)\}$$

$\rightarrow$ G has a Hamiltonian path iff $G \vee K_1$ has a Hamiltonian cycle

G has H.P. $\Rightarrow$ $G \vee K_1$ has a H.C.

so we can construct



→ we can construct a H.C. on $G \vee K_1$ through our K_1 and start/end verts of the H.P.

$G \vee K_1$ has a H.C. $\Rightarrow$ G has a H.P.

Note: ~~at most~~ exactly 2 of K_1's added edges are part of the H.C.

- w.l.o.g assume these edges attach to some u, v
- if we delete K_1, we still have a u,v-H.P. $\square$

We can reconsider Chvátal's condition for Hamiltonian paths

If G has degrees

$d_1 \leq d_2 \leq \ldots \leq d_n$

Then

Then

$i < \frac{n+1}{2}$ implies $d_i \geq i$

or $d_{n+1-i} \geq n-i$

$\Rightarrow G$ has a Hamiltonian Path