More definitions:

<u>complement</u> of $G \to \bar{G}$

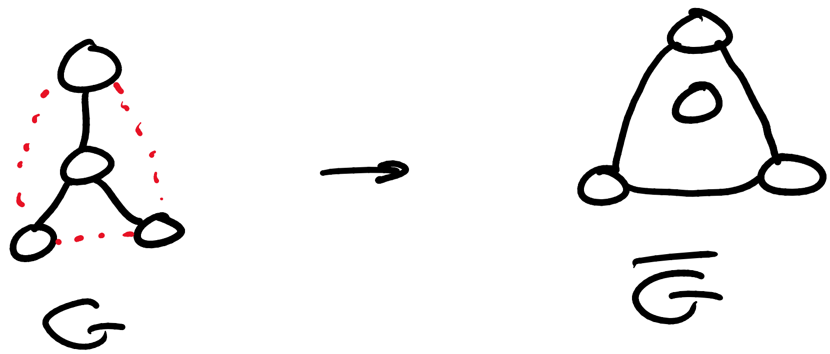
(assume G is simple)

such that

$V(\bar{G}) = V(G)$

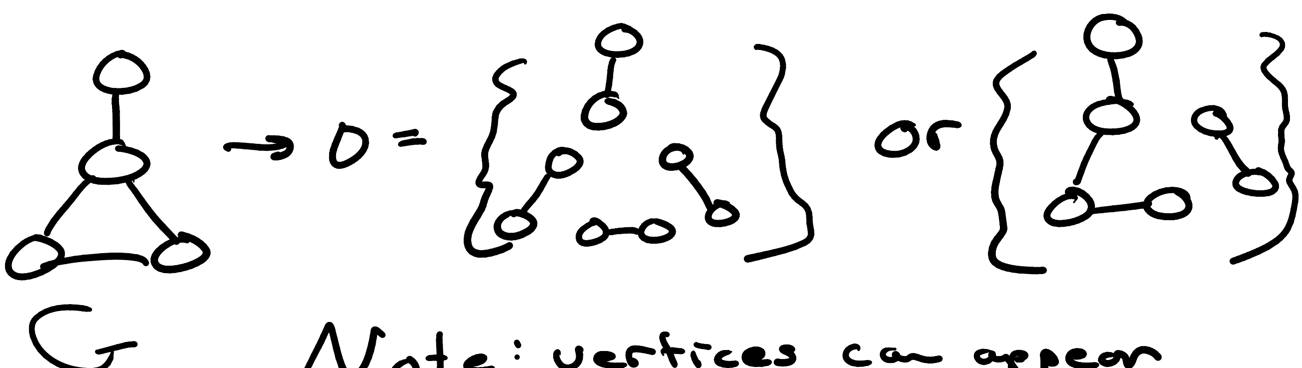
$E(\bar{G}) = \{\forall v, u \in V(G) : (u, v) \notin E(G)\}$

not in



<u>decomposition</u> of G

set of non-induced subgraphs s.t. each edge of G appears exactly once within this set



Note: vertices can appear any number of times, while edges

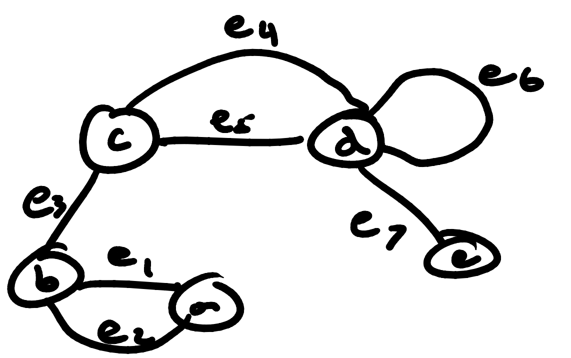
any number of times, while
edges appear exactly once

Time to take a stroll

walk: a list of vertices and edges
s.t. each listing is adjacent
or incident to the listing
preceeding and proceeding it

Trail: as above, but edges don't repeat

Path: as above, but edges and
vertices don't repeat



$W: \{a\, e_1\, b\, e_3\, c\, e_3\, b\, e_3\, c\, e_4\, d\, e_6\, d\}$

$T: \{a\, e_2\, b\, e_3\, c\, e_5\, d\, e_6\, d\}$

$P: \{a\, e_1\, b\, e_3\, c\, e_5\, d\}$

W is an a,d-walk

T is an a,d-trail

P is an a,d-path

A walk/trail/path that starts and ends at the same vertex is <u>closed</u>

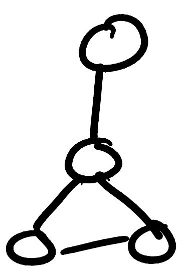
Note: a closed path is actually a cycle

<u>Length</u>: number of traversed edges

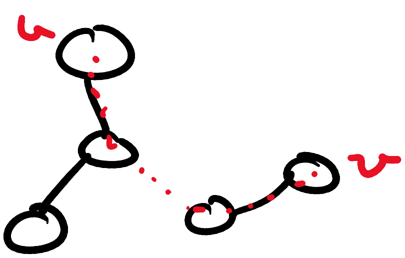
<u>Hop</u>: traverse of a single edge

Let's get connected

Recall: G is connected if
$\forall u,v \in V(G): \exists\ u,v\text{-path}$



G is connected



G' is disconnected

<u>connected component</u>: a <u>maximal</u> connected subgraph of G

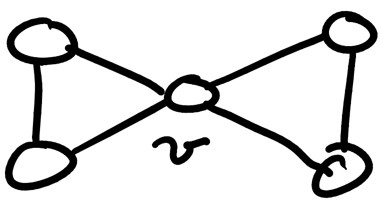
<u>maximal</u>: can't be made larger

<u>maximum</u>: largest possible

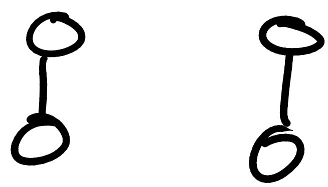
Note: same for minimal/minimum
but smaller/smallest

<u>cut vertex</u>: some $v \in V(G)$ s.t.
$G - v$ has more connected components (ccs)

remove v and all incident edges



G

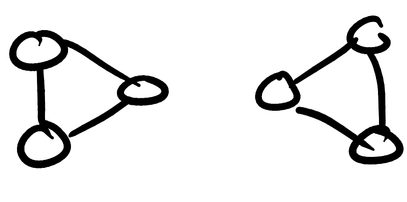


$G - v$

<u>cut edge</u>: some $e \in E(G)$ s.t.
$G - e$ has more components

Note: endpoints of e remain

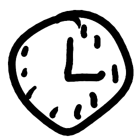
G



$G - e$

G   G - e

Time for the meat of graph theory

 4 

weak induction

Prove: $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$

Basis: $P(1) \Rightarrow 2^1 = 2^2 - 2 = 2$ ✓

Inductive Step: $P(n = k+1)$

Inductive Hypothesis: we assume that what we're trying to prove holds for $P(k)$

Show: $P(k+1)$ holds

$P(n = k+1) = 2^1 + 2^2 + \ldots + 2^k + 2^{k+1}$

If we assume $P(k)$ holds

$P(n) = 2^{k+1} - 2 + 2^{k+1}$

$= n+1$ ... $n+1$

$P(n) = 2^{\ldots} - 2 + 2^{\ldots}$

$P(n) = 2^{k+2 \,=\, n+1} - 2 = 2^{n+1} - 2 \quad \square$

weak induction

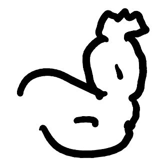
$P(1), P(2), \ldots, P(k), P(k+1), \ldots, \infty$

↑ Basis

↑ assume holds via I.H.

↑ show it still holds

alternatively...

strong induction  ← arm

$P(1), \ldots, P(k), \ldots, P(n)$

↑ basis

↑ assume for *all* $1 \le k < n$

↑ show it holds

Example proof:

show every closed odd walk ← length is odd
contains an odd cycle

Induction on the length of the walk

Basis: $P(1)$:  ✓

Basis: P(1):  v

Inductive step: $P(n > k \geq 1)$

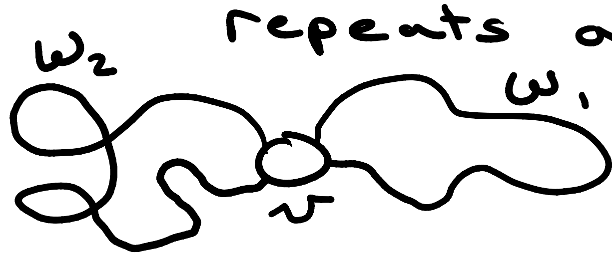
Assume we have a walk of length n, n = odd, walk is closed

Consider the cases

Case 1: no vertices repeat on our walk

=> our walk is just a cycle

Case 2: at least one vertex v repeats on our walk



this implies $w = w_1 + w_2$

integer parity

this means odd + odd = even

even + even = even

odd + even = odd

this implies w.l.o.g. $|w_1|$ = odd

↑ without loss of generality

without loss of generality

Note: $|W_1| < |W|$

so we can define $P(k) = W_1$

and invoke our I. H. on $P(k)$
inductive hypothesis

→ this tells us that there exists some odd cycle on $P(k)$

Now lets bring it on back

to do so, we need to show our property still holds on $P(n)$

Pretty easy in this case:

adding W_2 back to W_1 to create $P(n) = W$ will not delete that odd cycle on W_1

$\Rightarrow \exists$ odd cycle on W ☑

Necessity and sufficiency

Necessity and sufficiency
aka equivalence relations

Graph with property A is equivalent
to graph with property B

G with A $\Leftrightarrow$ G with B

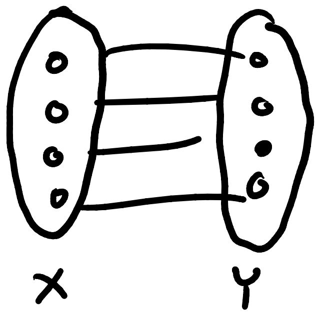
A is a necessary condition for B
G with A $\Rightarrow$ G with B

B is a sufficient condition for A
G with B $\Rightarrow$ G with A

To prove an equivalence A $\Leftrightarrow$ B
prove A $\Rightarrow$ B
prove B $\Rightarrow$ A

Prove: G has no odd cycles
$\Leftrightarrow$ G is bipartite

First show: $\Leftarrow$



bipartition of G

- Note: cycle is a closed path
- any path of G wlog starts at some $v \in X$, goes to $u \in Y$, to some $w \in X$, etc.

→ any odd path from X finishes in Y

⇒ no closed odd path can exist

We will prove the other direction on Monday

Also: Q2 due Monday @ midnight