Lecture 4 - Euler Tours and Degrees

Monday, January 23, 2023 1:19 PM

Now let's finish up our proof from last class

We showed bipartite => no odd cycles

Now: no odd cycles => bipartite

w.l.o.g. assume G is connected (as we can apply this argument to each of G's component)

consider some $v \in V(G)$

define $f(u): u \in V(G)$

$f(u)$ = shortest path distance to vertex v from u

$$A = \{ \forall a \in V(G) : f(a) = \text{even} \}$$

$$B = \{ \forall b \in V(G) : f(b) = \text{odd} \}$$

Q: are A and B independent?

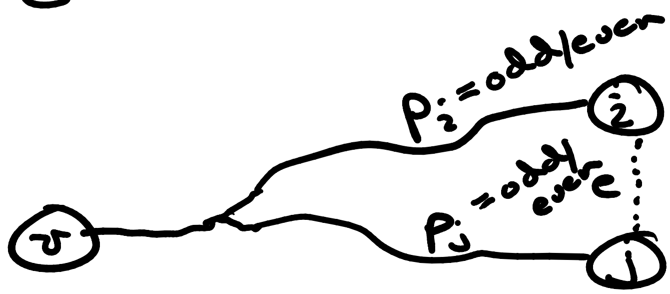
→ no edges among A vertices to A vertice or B to B

Note: $A \cap B = \phi$

or B to B

Consider two vertices in either A or B, $i, j \in A$ <u>or</u> $i, j \in B$

Consider shortest v,i-path v,j-path



Q: can edge e exist?

Note: p_i and p_j are both odd or both even

consider a walk from $v \to i$ then across hypothetical edge (i,j) then from $j \to v$

We already proved that all odd walks contain an odd cyle

Since any walk as described above will be odd

$\Rightarrow$ existence of edge (i,j)

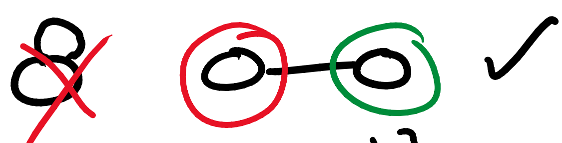
$\Rightarrow$ existence of edge (i, j) implies the existence of an odd cycle

X Contradiction X

$\Rightarrow$ so no such edge can exist, and A, B are a valid bipartition of G $\square$

Now let's show the same as an inductive proof

- We'll do induction on $E(G)$

Basis $P(1)$: 

bipartite

Consider $P(n)$ with no odd cycle

construct $P(k) = P(n) - e \; : e \in E(P(n))$

Note: edge deletion does not create cycles

I.H. on $P(k)$ gives us a valid bipartitioning of $V(P(k))$

Now: add back that edge e
and show whether the bipartitioning
is still valid

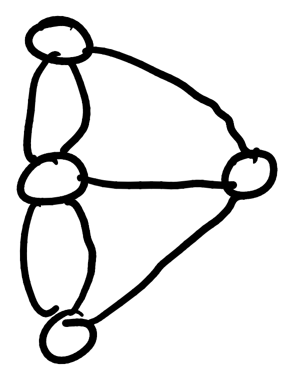
Case 1: e goes between vertices in
separate bipartite sets

→ P(n) has a valid bipartitioning

Case 2: e goes between vertices
in the same sets

→ you show whether
this is possible

Recall: Euler and the bridges
of Königsberg

as a graph 

Euler: does a closed trail exist
that traverses all edges?

that traverses all edges?
aka Euler Tour/Circuit

First: show if $\forall v \in V(G): d(v) \geq 2$

$$\Rightarrow \exists C_n \subseteq G$$

subgraph

To do this: we'll construct an

extremal argument

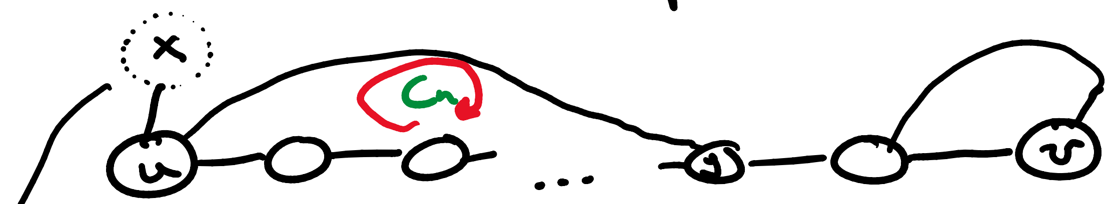
using the extremal principle

Extremal principle: with some set of
countable/orderable values

$\exists$ some item with a maximum
value and some item with a
minimum vale

For our proof, select $P_n \subseteq G$, where
P_n is a path of maximum length

What does that path look like?



consider that $\forall u \in V(G): d(u) \geq 2$

→ can such an x exist?

No, that would imply the existence of a longer path

⇒ so u has an adjacency with P_n, which would create a cycle C_n □

Eulerian graphs: graphs with an Euler Tour

Note: we already saw that connectedness and all degrees being even are necessary conditions for an Eulerian graph

→ Are these properties also sufficient?

$\to$ Are these properties
also sufficient?

if $\forall v \in V(G): d(v) = \text{even}$ and G is
connected $\Rightarrow \exists$ an Euler tour on G

We'll do induction on $|E(G)|$

Basis $P(0)$: $0 \to$ trivial tour $\{\}$

Assume we have $P(n)$ with all
even degrees and is connected

Note: minimum degree within
$P(n)$ is 2

From before: this implies the
existence of some $C \subseteq P(n)$

$P(k) = P(n) - C$

Note: deleting a cycle subtracts
exactly 2 degree from
every vertex on that cycle

Note x2: $P(k)$ might be
disconnected

However: our assumptions still hold for all components of $P(k)$

→ we can invoke our I.H. on all components of $P(k)$

→ this gives us Euler tours for each component

Q: how can we get back to $P(n)$?

PROOF B4 ALGOR-ITHM

to complete our proof:
combine the existing tours with C to get a tour on $P(n)$

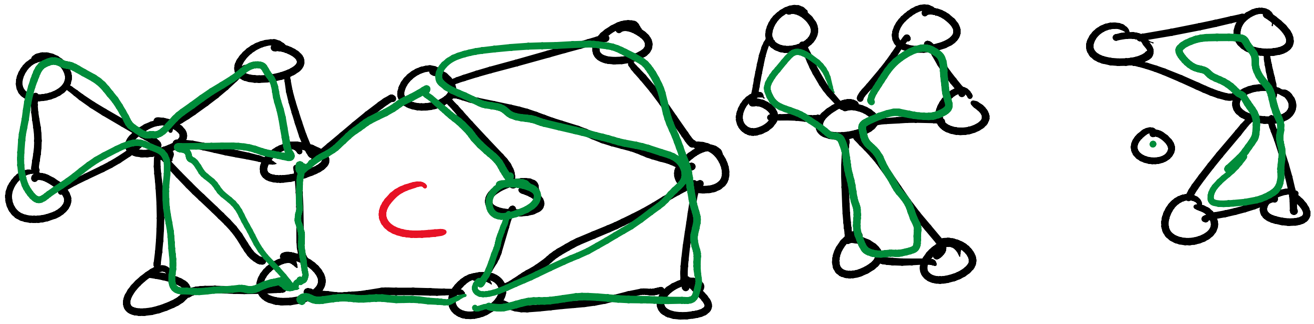
Our algorithm:

start on some $v \in C$
if $d(v) = 2$ continue on
else $\exists$ a tour from v into a component of $P(k)$
follow that tour

follow that tour

continue along C

→ Output: Euler tour on $P(n)$ □



$P(n)$ $\qquad\qquad$ $P(k) = P(n) - C$