Lecture 5 - Directed graphs and degrees

Wednesday, January 25, 2023 5:21 PM

Degrees

say $n = |V(G)|$, $m = |E(G)|$

degree of $v \to d(v)$, d_v

for simple graphs $d(v) = |N(v)|$

For graph G:

maximum degree $\to \Delta(G)$

minimum degree $\to \delta(G)$

G is k-regular if

$\delta(G) = k = \Delta(G)$

$\forall v \in V(G): d(v) = k$

Note: all C_n are 2-regular

all K_n are $(n-1)$-regular

Degree sum formula:

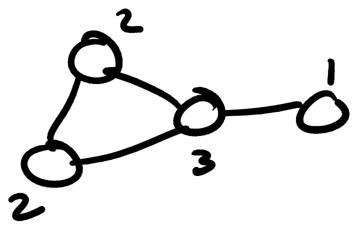
$$\sum_{v \in V(G)} d(v) = 2m \leftarrow \text{even}$$

$$\sum_{v \in V(G)} d(v) = 2m$$

why: each edge adds +1 degree to each endpoint vertex

Q: what sets of possible degrees are valid for some graph?

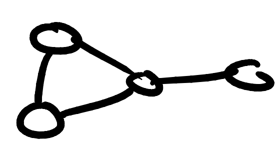
Degree sequence: list of all degrees for all vertizes in some graph

G  $S = \{1, 2, 2, 3\}$

Graphic sequence: a list of degrees that can realize a simple undirected

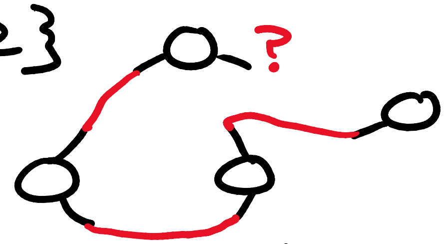
realize: a graph can be constructed with those degrees

$S = \{1, 2, 2, 3\}$

↳ realize 

S is graphic

$S' = \{1,2,2,2\}$



sum is odd

can't realize

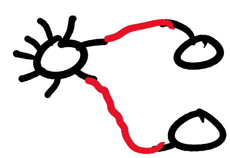
S' is **NOT** graphic

Note: an even degree sum is a <u>necessary</u> condition for a graph sequence

Q: is it sufficient?

Proof by counter-example

$S = \{8, 1, 1\}$



Q: How can well tell if a sequence is graphic?

sequence is graphic?

Big dog of GT

Havel-Hakimi Theorem

Given non-increasing sequence $S = \{d_1\ d_2\ d_3 \dots d_n\}$

$$d_1 \geq d_2 \geq d_3 \geq \dots \geq d_n$$

S is graphic iff sequence

$$S' = \{(d_2-1)\ (d_3-1) \dots (d_{(d_1+1)}-1) \dots d_n\}$$

is graphic

Example: $S = \{\cancel{3}, 2, 2, 1\}$

$-1 \quad -1 \quad -1$

$S' = \{\cancel{1}\ 1\ 0\}$

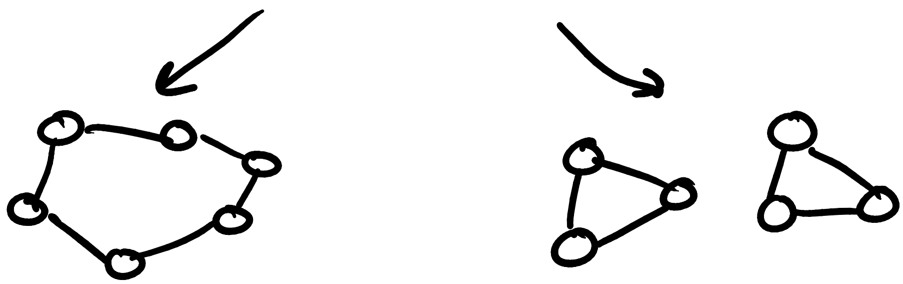
-1

$s'' = \{0\ 0\}$

$s'' = \{0\ 0\}$

Note 1: a graphic sequence does not necessarily have a unique realization

consider $S = \{2, 2, 2, 2, 2, 2\}$



Note 2: We are able to realize a graph with a given sequence using this process

→ Havel-Hakimi Algorithm

1. Map each value in S to a vertex
2. We draw edge $(d_i d_j)$ when some d_j is decremented by the removal of d_i to

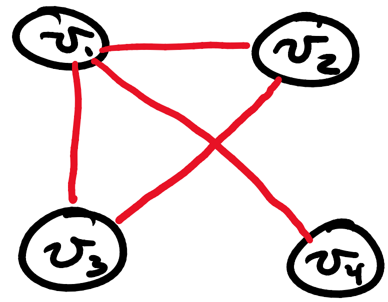
by the removal of d_i to create S' from S

3. Iterate until S^* is empty

$$\begin{array}{ccccc} & v_1 & v_2 & v_3 & v_4 \\ S = & \{\cancel{3}, & 2, & 2, & 1\} \\ & & -1 & -1 & -1 \end{array}$$

$$\begin{array}{cccc} S' = & \{\cancel{1}, & 1, & 0\} \\ & & -1 & \end{array}$$

$$S'' = \{0\ 0\}$$

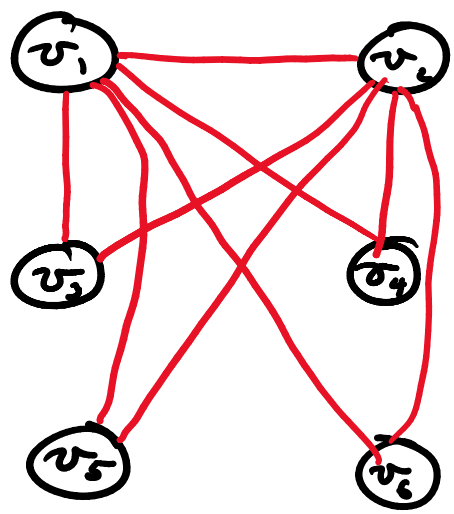


Bigger example:

$$\begin{array}{ccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ S = & \{\cancel{5} & 5 & 2 & 2 & 2 & 2\} \\ & & -1 & -1 & -1 & -1 & -1 \end{array}$$

$$\begin{array}{cccccc} S' = & \{\cancel{4} & 1 & 1 & 1 & 1\} \\ & & -1 & -1 & -1 & -1 \end{array}$$

$$S'' = \{0\ 0\ 0\ 0\}$$



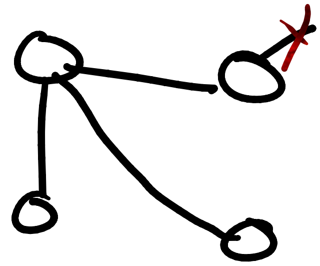
More examples:

$S = \{\cancel{6}\ \underset{-1}{5}\ \underset{-1}{4}\ \underset{-1}{3}\ \underset{-1}{2}\}\ \ -1\ \ -1$

$S' = ?$

Not graphic

$S = \{\cancel{3}, \underset{-1}{2}, \underset{-1}{1}, \underset{-1}{1}\}$

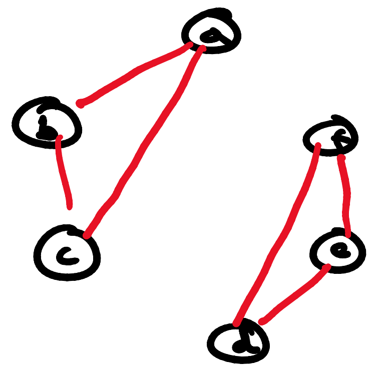


$S' = \{1, 0, 0\}$

Not graphic

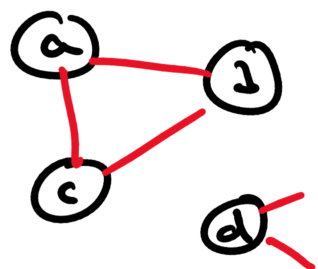
Consider:

$S = \{\overset{a}{\cancel{2}}\ \ \overset{b}{\underset{-1}{2}}\ \ \overset{c}{\underset{-1}{2}}\ \ \overset{d}{2}\ \ \overset{e}{2}\ \ \overset{f}{2}\}$



$S' = \{\overset{b}{\cancel{1}}\ \ \overset{c}{\underset{-1}{1}}\ \ \overset{d}{2}\ \ \overset{e}{2}\ \ \overset{f}{2}\}$

$S'' = \{2\ 2\ 2\}$



$S = \{\overset{a}{\cancel{2}}\ \ \overset{b}{\underset{-1}{2}}\ \ \overset{c}{\underset{-1}{2}}\ \ \overset{d}{2}\}$

$S' = \{\cancel{1}\ \ \underset{-1}{\cancel{1}}\ \ 2\}$

$S'' = \{2\}$

Note: it is necessary to sort

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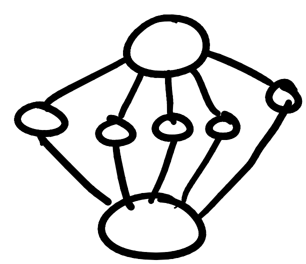
Braincercize: can all possible realization for a given graphic sequence be constructed via Havel-Hakimi?

→ $\boxed{\text{No}}$

$S = \{5\ 5\ 2\ 2\ 2\ 2\ 2\}$



However: we can generate some subset of possible configurations by permuting equal value in the sequence

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Directed graphs

aka digraphs

(also known as)

→ we consider directionality for every edge

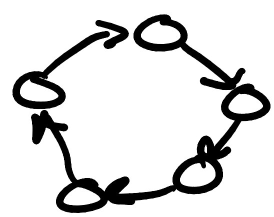
$e = (u, v)$

tail   head
predecessor   successor

for walks, trails, paths, cycles

→ same definitions as with undirected graphs, but now they follow edge direction

directed cycle



directed path

We also have the notion of simple, loopy, and multi digraphs

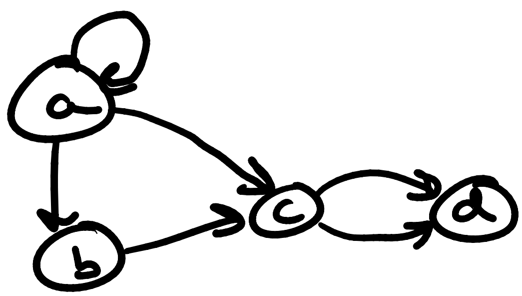
Note: a digraph is still simple if it has edges e, f

if it has edges e, f
where $e = (u,v)$, $f = (v,u)$

simple　　　loopy　　　multi

Adjacency matrix

- No longer symmetric
- a nonzero at (i,j) implies directed edge from $i \to j$



$$
\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}
\begin{array}{c}
\begin{array}{cccc} a & b & c & d \end{array} \\
\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

$\sum row_i$ = out degree of i

$\sum col_i$ = in degree of i

out degree $d^+(v)$ = # edges coming out of v

in degree $d^-(v)$ = # edges going into v

$N^+(v)$ = out neighborhood
vertices that have edges from v

$N^-(v)$ = in neighborhood
vertices that have edges to v

$\Delta^+(G), \Delta^-(G)$ = max out and in degrees on G

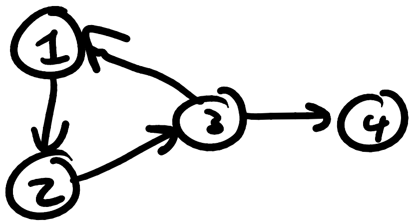
$\delta^+(G), \delta^-(G)$ = min out and in degrees on G

Degree sum formula

$$\sum_{v \in V(G)} d^+(v) = \sum_{v \in V(G)} d^-(v) = m$$

Degree sequences

Note: we must now consider (in, out) degree pairs (in, out) degree pairs



$S = \{(1,1), (1,1), (1,2), (1,0)\}$

Q: How can we tell if S is graphic?

Necessary condition:
sum of the out degrees must equal the sum of the in degrees

Q: Is this condition also sufficient?
Generally for simple digraphs
$\longrightarrow$ No
For loopy multi-digraphs
$\rightarrow$ Yes

PROOF BY ALGORITHM

Consider $(d_i^+, d_i^-) : 1 \leq i \leq n$

$$m = \sum_{1 \le i \le n} d_i^+ = \sum_{1 \le i \le n} d_i^-$$

consider m lines

d_i^+ dots get labeled i

d_i^- dots get labeled $-i$

consider n vertizes $1 \ldots n$

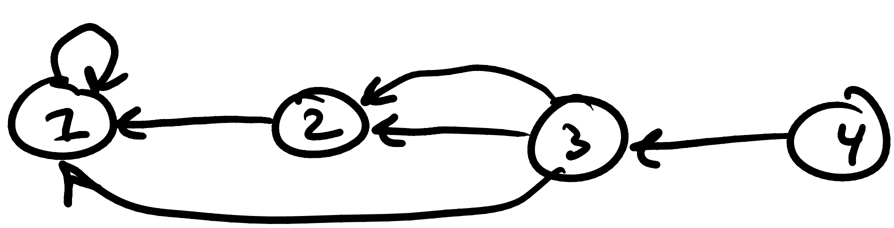
$v_1 \ldots v_n$

For each line

construct an edge as a positive label to a negative label corresponding to the vertex set

$$S = \{\overset{i=1}{(1,3)}, \overset{i=2}{(1,2)}, \overset{i=3}{(3,1)}, \overset{i=4}{(1,0)}\}$$

$$\begin{array}{cccccc} 1 & 2 & 3 & 3 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ -1 & -1 & -1 & -2 & -2 & -3 \end{array}$$



Eulerian Digraphs

A digraph is Eulerian iff there exists a closed trail containing all edges

Proof: same as with the undirect case