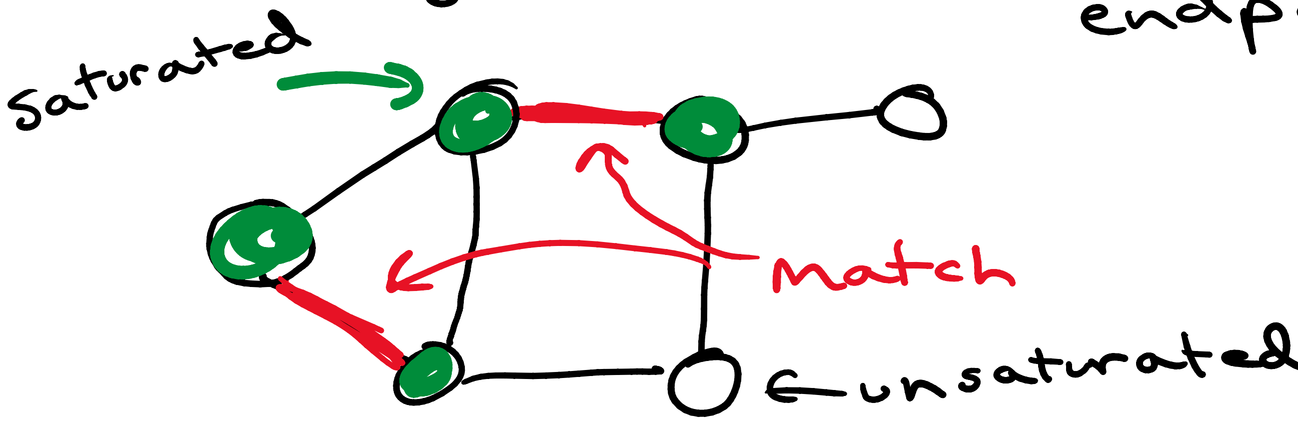
Lecture 10 - Bipartite Matching

Saturday, February 11, 2023 5:54 PM

<u>Match</u> on G - a set of non-loop edges with no shared endpoints

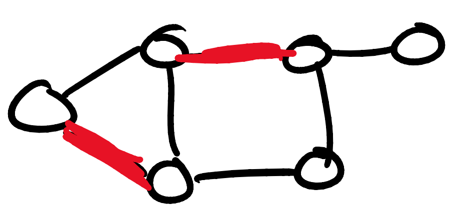


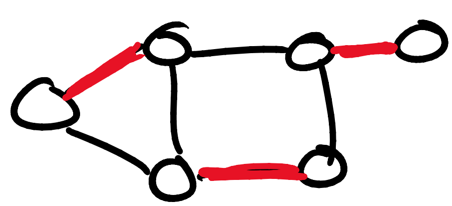
saturated vertex - vertex with an incident matched edge

unsaturated vertex - vertex with no incident matched edge

maximum match - largest possible match on same graph

maximal match - a match that can't be made larger


maximal


maximum

perfect match - a match that saturates all vertices

$$|M_p| = \frac{|V(G)|}{2} \quad M_p = \{e_i \dots e_j\}$$
$$e_e \in E(G)$$

Does G have a perfect match?

Necessary conditions:

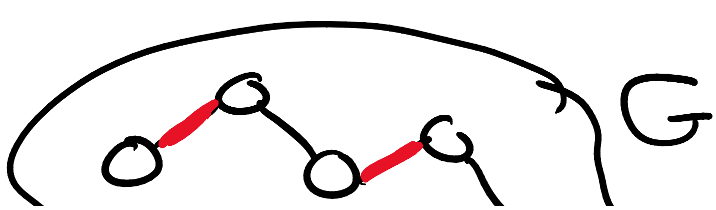
→ $|V(G)|$ must be even

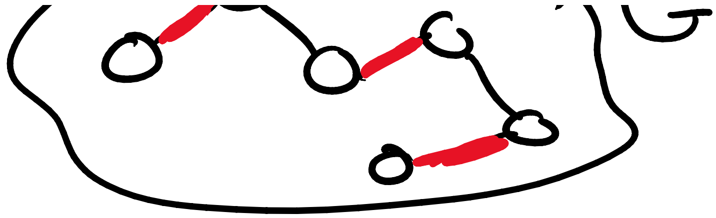
→ $d(v) > 0 : \forall v \in V(G)$

sufficient conditions?

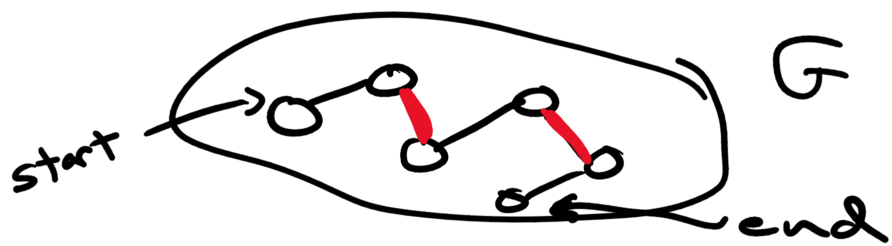
A lot tougher to say

M-alternating path - given a match M on same G, an M-alt. path is a path subgraph that alternates matched and unmatched edges

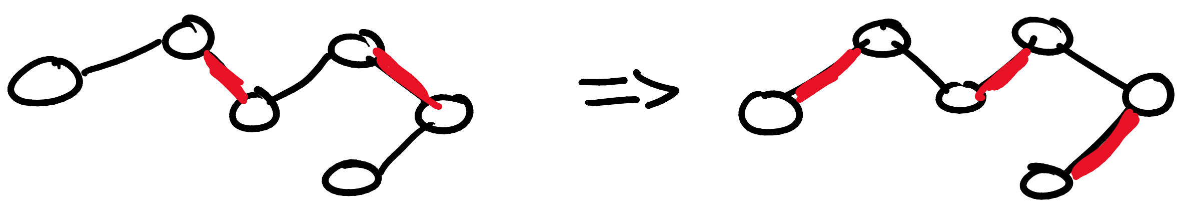




M-augmenting path - an M-alt. path that starts and ends on unmatched vertices



Note: along an M-aug. path, we can swap matched edges to create a larger match



=> This implies that if G has an M-aug path, then M is not maximum

Q: does every non-maximum match

Q: does every non-maximum match have an M-aug path?

Berge: a matching M on G is maximum iff there exists no M-aug paths

★ **Contrapositive** ★

contrapositive: $P \to Q$

$\neg Q \to \neg P$

or: $P \Leftrightarrow Q$

$\neg P \Leftrightarrow \neg Q$

Contrapositive of Berge:

M is not maximum $\Leftrightarrow$ there exists an M-aug path

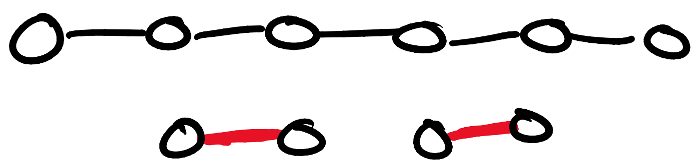
($\Leftarrow$) We already showed how to increase a match M given an increase a match M given an M-aug. path P

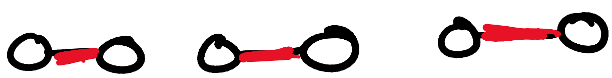
→ take the <u>symmetric difference</u>

↳ aka an XOR

P

M



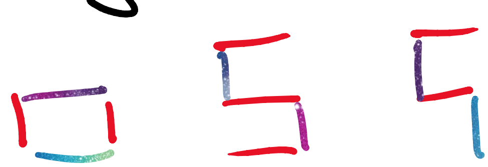
$P \Delta M =$  $= M'$, our new larger match

↑ symmetric difference

($\Rightarrow$) consider M as our current match and M' as a larger match where $|M'| > |M|$

consider $F = M' \Delta M$

Note: F is comprised of even cycle and paths



Consider only odd paths in F

→ we know at least one must exist as $|M'| > |M|$

$\Rightarrow$ this is our M-aug. path $\square$

Maximum Bipartite Matching

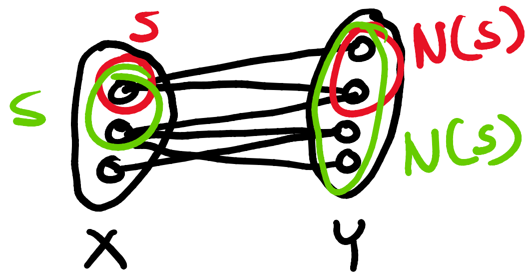
→ we consider optimality with respect to the "small" bipartite set

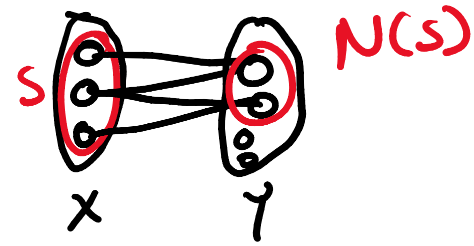
so for $B \Rightarrow V(B) = \{X, Y\}$

$$|X| \leq |Y|$$

→ if $\exists M$ that saturates X, then M is maximum

Hall: $\exists M$ that fully saturates X where $|X| \leq |Y|$ on X,Y-bigraph iff $\forall S \subseteq X : |N(S)| \geq |S|$

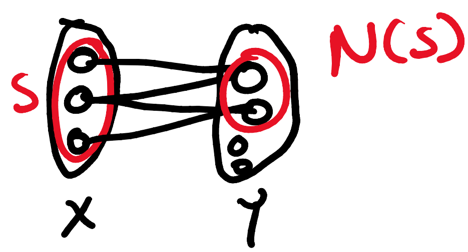




contrapositive

if $\exists\, S \subseteq X$ s.t. $|N(S)| < |S| \iff$ no fully X-saturating match

($\Rightarrow$) we just showed this above



No way to fully match S to a smaller N(S)

($\Leftarrow$)
- consider some maximum M
- consider some $u \in X$, $u \notin M$
- consider S = all $v \in X$ that are reachable via a u,v-M-alt path

reachable via a u,v-M-alt path

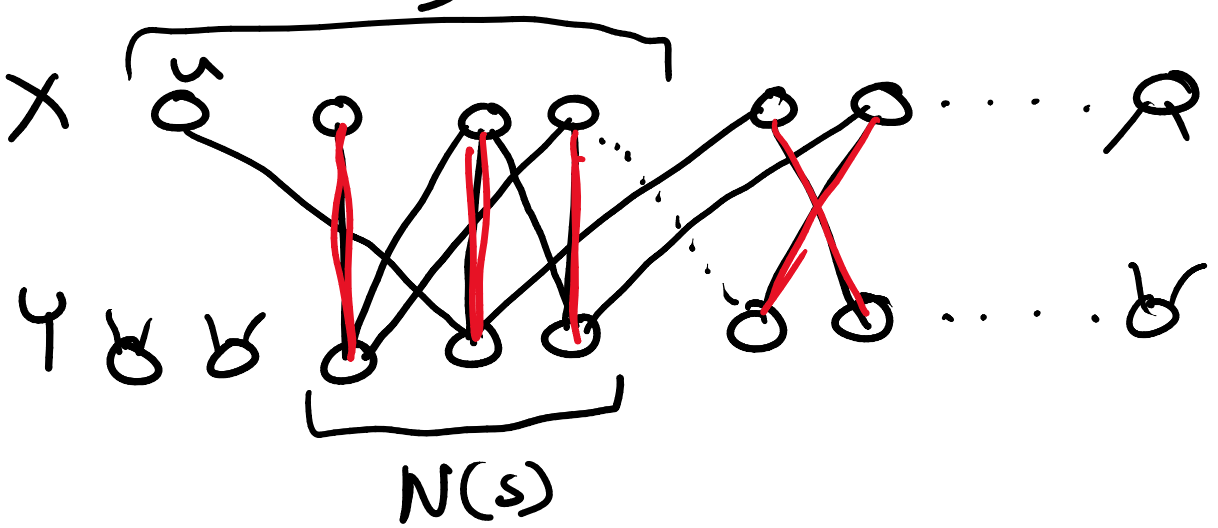
Note: $N(S)$ is fully saturated

→ if not, we have an M-aug path

Finally: considering a bijection from $N(S) \leftrightarrow S-u$, we have

$$|N(S)| = |S| - 1 < |S|$$

$$|N(S)| < |S| \quad \square$$



Maximum bipartite matching via algorithm

we have bipartite G with match M

with match M