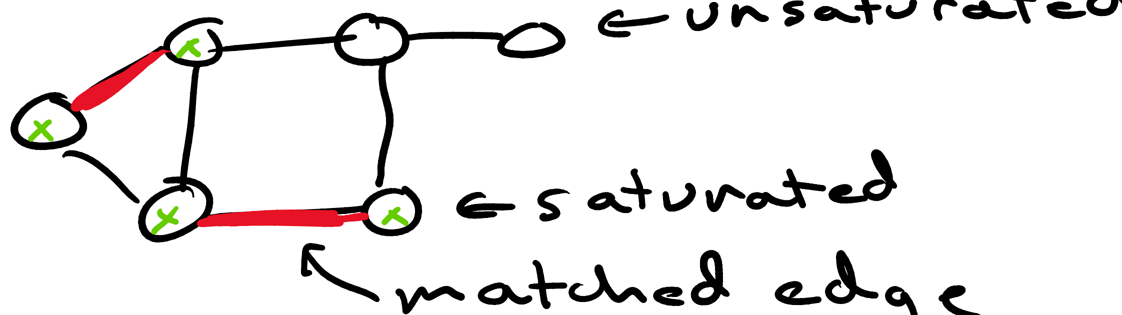
Lecture 11 - General Matching and Covers

Wednesday, February 15, 2023 6:34 PM

Review

Match: set of edges with no shared endpoints

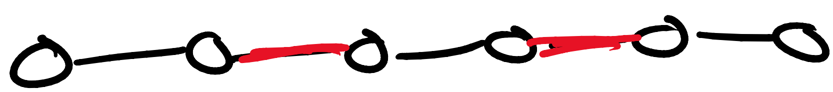


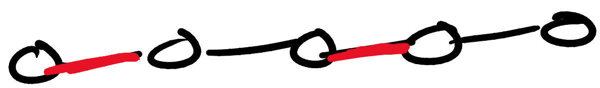
Maximum: largest possible match

Maximal: can't be made large

Perfect: saturate all vertices

Berge: M is maximum of G iff G has no M-aug path

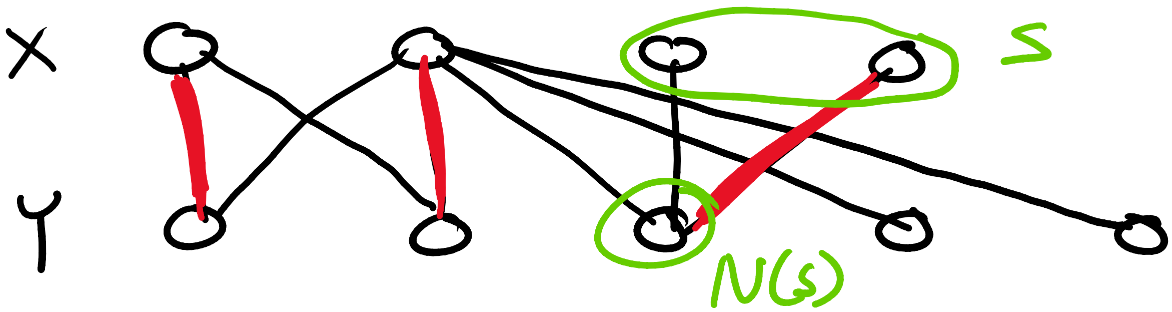
M-aug 

M-alt 

Hall: $\exists M$ that saturates X in an X,Y-bigraph $|X| \le |Y|$

an X,Y-bigraph $|X| \le |Y|$

iff $\forall S \subseteq X \ |N(S)| \ge |S|$



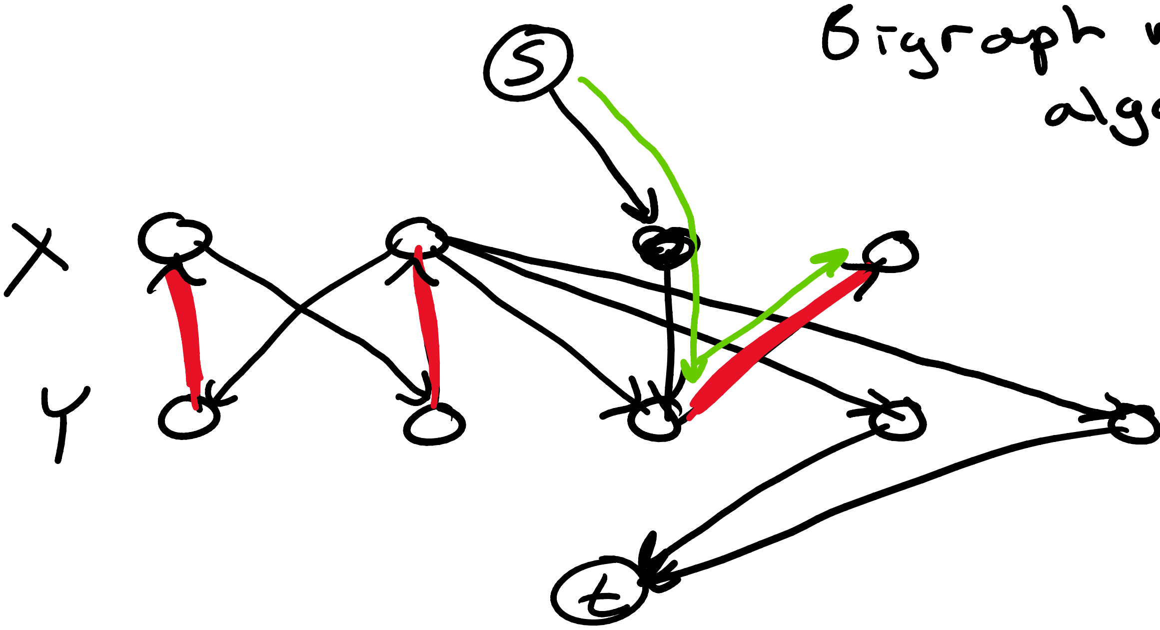
Is this match optimal? Yes

★ Contrapositive ★ of Hall

No X-saturating match

iff $\exists S \subseteq X \ |S| > |N(S)|$

Bigraph match algorithm



General Graph Matching

General Graph Matching

$o(G)$ = # of odd components of graph G

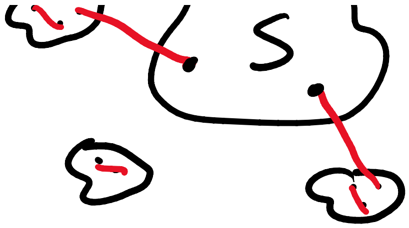
Tutte: G has a perfect match iff $\forall S \subseteq V(G): o(G-S) \leq |S|$

G has P.M. $\Rightarrow \forall S \subseteq V(G): o(G-S) \leq |S|$

- consider some S
- consider some $G-S$

Note: each odd component cannot be perfectly matched

→ at least one vertex in each component must be matched to some vertex in S



$\Rightarrow$ so $|S|$ must be bounded below by $o(G-S)$

$\forall S \subseteq V(G): o(G-S) \leq |S| \Rightarrow G$ has P.M.

contrapositive

G has no P.M. $\Rightarrow \exists S \subseteq V(G)$ s.t. $|S| < o(G-S)$

Note: condition holds if we add edges to G

Extreme

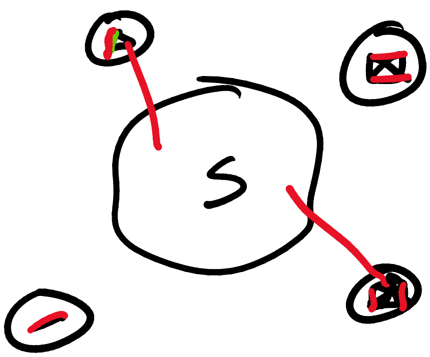
we consider an extremal choice of $G \to G'$, where G' is edge-maximal with respect to having no P.M.

no P.M.

$\hookrightarrow G' + e$ has P.M.

Define $S = \{\forall v \in V(G) : d(v) = |V(G)| - 1\}$

Case 1: $G' - S \rightarrow$ all components are cliques

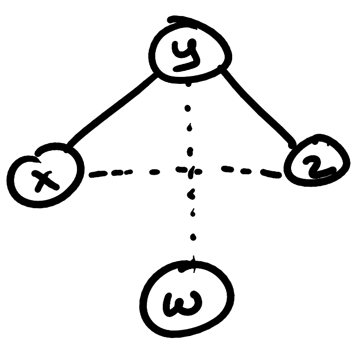


Note: S must be "bad"

$|S| < o(G-S)$

otherwise we can construct a P.M. on G

Case 2: $G' - S \rightarrow$ not all cliques



$\exists\, x, z$ s.t. $(x,z) \notin E(G'-S)$

$\exists\, y$ s.t. $(x,y)(z,y) \in E(G'-S)$

$\exists\, w$ s.t. $(y,w) \notin E(G'-S)$

We know adding $e = (x,z)$ or $e = (y,w)$ creates a P.M. of $G' + e$

We'll show that this also implies

We'll show that this also implies a P.M. on G itself

- define

M_1 = P.M. on $G' + (x,z)$

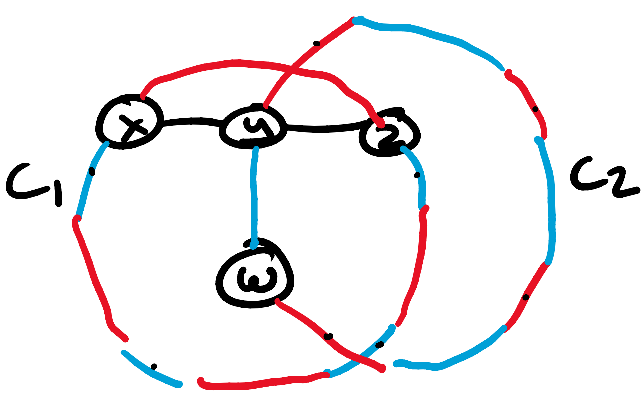
M_2 = P.M. on $G' + (y,w)$

$F = M_1 \Delta M_2 \rightarrow$ must be paths or cycles

M_1:

M_2:

However: these are both P.M., so we only have cycles



C_1 = cycle with (x,z)

C_2 = cycle with (y,w)

Case 2a: $C_1 \neq C_2$

P.M. on G' = all $e \in M_2$, $e \in C_1$

all other $e \in M_1$

↳ P.M. w/o (x,z) or (y,w)

x contradiction x

P.M. w/o (x,z) or (y,w)

contradiction

=> S must be "bad"

Case 2b: $C_1 = C_2$

P.M. on $G' = M_1$ on C_2 from w until x or z

if we reach x:

- P.M. on $G' += (x,y) + M_2$ from y to z

if we reach z:

- P.M. on $G += (y,z) + M_2$ from y to x

either way we have a P.M. w/o (x,z) or (y,w)

Contradiction

so case 2 can't exist

so S must be bad

so case 2 can't exist.

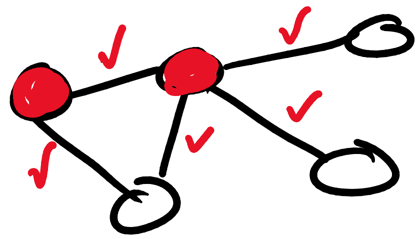
so S must be bad

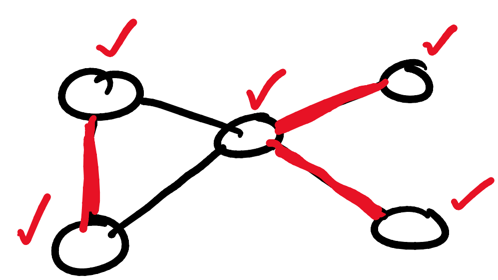
$|S| < o(G-S)$ $\square$

$\boxed{\text{Tutte: } G \text{ has P.M.} \iff \forall S \subseteq V(G): o(G-S) \leq |S|}$

Vertex cover: a set $Q \subseteq V(G)$ that has at least one endpoint for all $e \in E(G)$

Edge cover: a set $L \subseteq E(G)$ that has at least one edge incident on all $v \in V(G)$





König-Egerváry: on a bipartite graph $G \Rightarrow$ the size of a minimum vertex cover = the graph

minimum vertex cover = the size of a maximum match

$|M_{max}|$ = max match

$|C_{min}|$ = min cover

K.E.: $|M_{max}| = |C_{min}|$ on bigraph G

Note: $|C| \geq |M|$ for any cover and match

→ every matched edge needs to be covered by at least one $v \in C$

This problem is a good example of a min-max relation

aka dual optimization problems

Solution to min. problem is upper bound for solution to max. problem

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$S \subseteq V(G)$ is a

Dominating set: $S \subseteq V(G)$ is a dominating set if

$$\forall v \in V(G): v \notin S \to \exists u \in N(v): u \in S$$

aka

every vertex is in S or has a neighbor in S