11.1 Matching in General Graphs

For the most part, we've discussed matching restricted to bipartite graphs. We're going to generalize it now to all graphs. First define the function $o(G)$ as the number of **odd connected components** in G. As we know, an odd connected component has an odd number of vertices. In order for a graph to have a perfect matching, we'll use what could loosely be considered as a generalization of Hall's Condition.

Tutte's Theorem states that a graph G with a perfect match satisfies the inequality $\forall S \subseteq V(G) : o(G - S) \leq |S|$. Formally, a graph $G = (V, E)$ has a perfect matching if and only if for every possible vertex set $S \subseteq V(G)$, the subgraph induced by $V - S$ has at most $|S|$ connected components with an odd number of vertices. Let's develop a proof for Tutte's Theorem.

11.2 Independent Sets, Dominating Sets, and Covers

A **vertex cover** of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint on all $e \in E(G)$. The vertices in Q *cover* $E(G)$. An **edge cover** of G is a set $L \subseteq E(G)$ such that L has at least one edge incident on all $v \in V(G)$. The edges in L *cover* $V(G)$.

The **König-Egerváry Theorem** states that if G is a bipartite graph, then the size of a maximum matching in G equals the minimum size of a vertex cover. We can prove this theorem too.

An **independent set** of vertices on a graph G are a set of vertices that are not connected by an edge. The size of a maximum independent set on G is called the **independence number** of G. For a bipartite graph, this isn't necessarily the size of the larger partite set.

In G, $S \subseteq V(G)$ is an independent set if and only if $\bar{S}$ is a vertex cover. Thus a maximum independent set is the complement of a minimum vertex cover, and their sizes summed equals the order of G. You'll be proving this in the third homework.

Similar to a vertex cover is the notion of **dominating sets**. A dominating set is some vertex set $S \subseteq V(G)$ where $\forall v \in V(G) : v \in S$ **or** $\exists u \in N(v)$ such that $u \in S$. That is, a set S is dominating on some G if all vertices of G are either in S or have a neighbor in S.