Lecture 12 - Connectivity

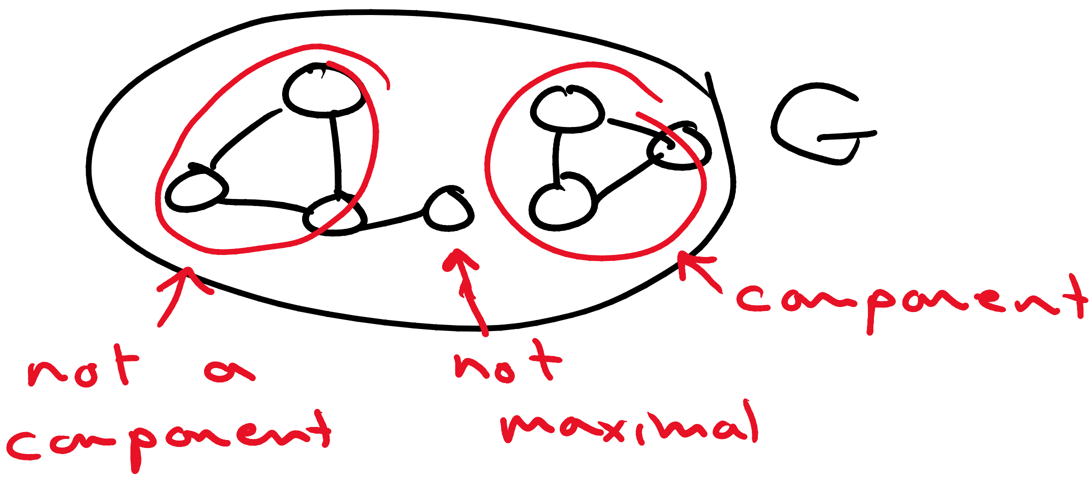
Saturday, February 18, 2023 12:34 PM

Review of connectivity

G is connected if

$\forall u,v \in V(G)$: $\exists$ u,v-path

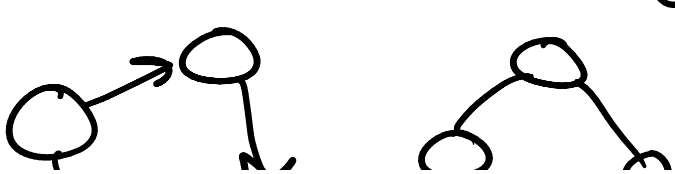
Connected component is a maximal connected subgraph



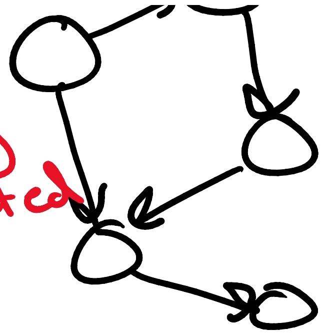
Weak connectivity

A digraph is weakly connected if the underlying graph is connected

graph if we ignore edge

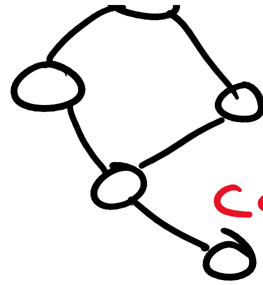


weakly connected



G

ignore edge directions

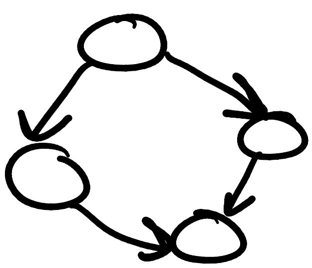


connected

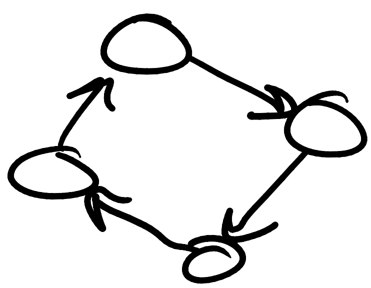
underlying graph

Strong Connectivity

A digraph is strongly connected if $\forall u,v \in V(G): \exists\, u,v$-path (directed path)

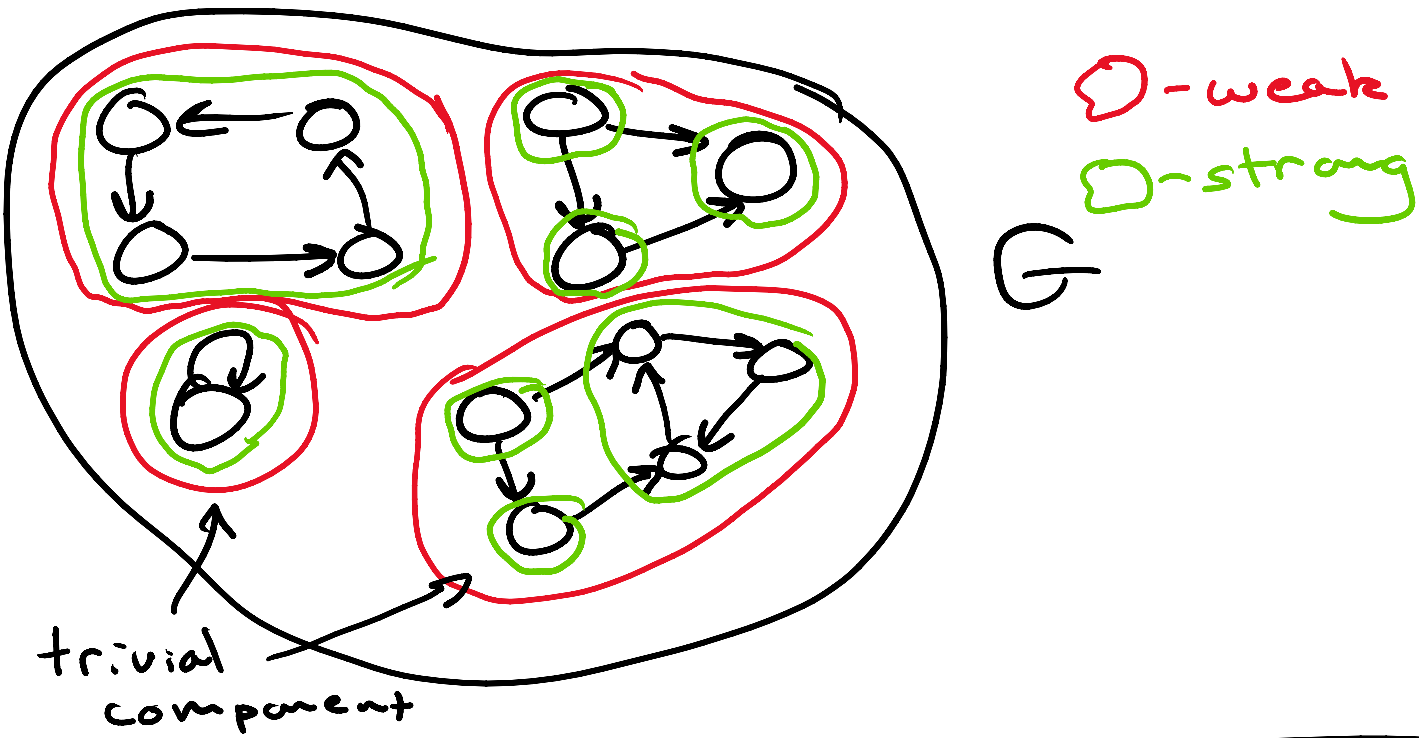


Not strongly connected



strongly connected

weak/strong components – a maximal weakly/strongly connected subgraph



Vertex Connectivity
(undirected graphs)

Recall: cut vertex is a vertex v in G s.t. $G-v$ has more components than G

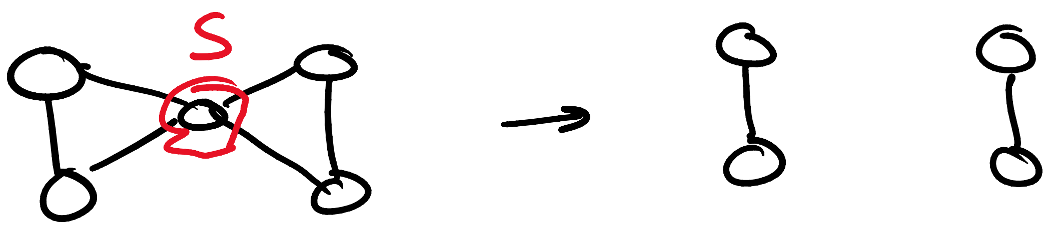
Separating set: a set $S \subseteq V(G)$ s.t. $G-S$ has more components than G

AKA: vertex cut
vertex separator

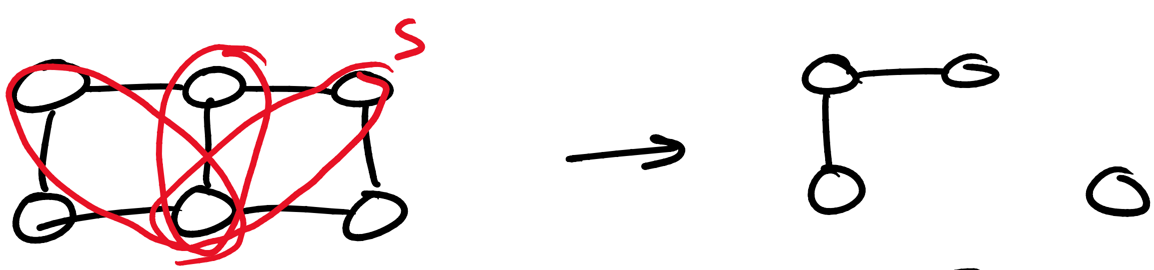
vertex separator

Connectivity of $G = \kappa(G) = k$
is the size of a minimum
vertex cut

G is k-connected if $k = \kappa(G)$



G $\kappa(G)=1$ $G-S$



G $\kappa(G)=2$ $G-S$

Note: for connectivity, the
maximum size of a separator
is $|V(G)|-1$

↳ K_n is $(n-1)$-connected

Also:

C_n is 2-connected

Tree T is 1-connected

T - any edge is <u>disconnected</u>

Edge connectivity

cut edge - an edge in some G s.t. $G-e$ has more components than G

Disconnecting set - a set of edges $F \subseteq E(G)$ s.t. $G-F$ has more comps. than G

AKA edge cut

Bond: minimal edge cut

↳ not necessarily minimum

<u>edge-connectivity</u> of $G = \kappa'(G) = k$

edge-connectivity of G

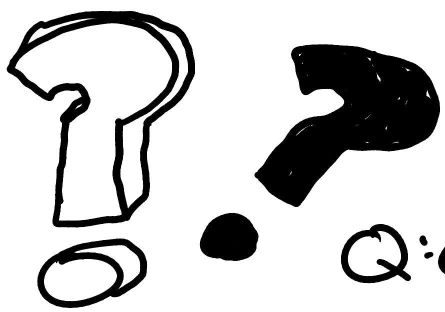
minimum size of an edge cut

→ G is k-edge-connected

Note: if G is k-connected or k-edge-connected then G is (k-1)-connected or (k-1)-edge-connected

---

Bounds on Connectivity



$$\kappa(G) \overset{?}{\underset{-}{<>}} \kappa'(G) \overset{?}{\underset{-}{<>}} \delta(G)$$

Q: Can we place bounds on these relationships?

Note: trivially, if we remove all edges incident on a minimum degree vertex in G, we will disconnect it

$$\kappa'(G) \leq \delta(G)$$

disconnect...

$\rightarrow \kappa'(G) \leq \delta(G)$

Likewise: we can remove all neighbors of that minimum degree vertex

$\rightarrow \kappa(G) \leq \delta(G)$

But what about $\kappa(G)$ and $\kappa'(G)$: how are they bounded?

Extreme

Consider a minimum edge cut F that separates G is $S, \bar{S}$ s.t. $\bar{S} = V(G) - S$

Case 1: $\forall u \in S, \forall v \in \bar{S} : \exists (u,v) \in E(G)$

$\rightarrow$ this implies

$$\kappa'(G) = |F| = |S||\bar{S}| \geq |V(G)| - 1 \geq \kappa(G)$$

Note: $\kappa(G) \leq |V(G)| - 1$

so $\rightarrow \kappa'(G) \geq \kappa(G)$

$\hookrightarrow$ so $\rightarrow \kappa'(G) \geq \kappa(G)$

Case 2: $\exists x \in S, \exists y \in \bar{S} : (x,y) \notin E(G)$

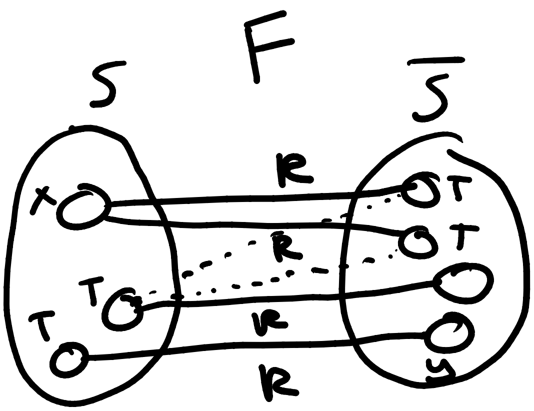
STRUCTURAL ARGUMENT

define $T =$ all $u \in N(x) : u \in \bar{S}$
and all $v \in S - x : \exists (v,z) \in E(G)$
$z \in \bar{S}$

Note: all x,y-paths must go through T

$\rightarrow T$ is a x,y-vertex cut

define $R =$ all $e \in (x,w) : w \in T \cap \bar{S}$
and $f = (a,b) : a \in T \cap S,\ b \in \bar{S}$
(where we only select one for each a)



Note: $|R| = |T|$

As we've selected only one f for each R out of multiple possible

$|R| \leq |F|$

$$|R| \le |F|$$

$$\Rightarrow |T| \le |F|$$

$$\boxed{\kappa(G) \le \kappa'(G)}$$

All together now 

$$\boxed{\kappa(G) \le \kappa'(G) \le \delta(G)}$$