Lecture 13 - 2-conncted Graphs

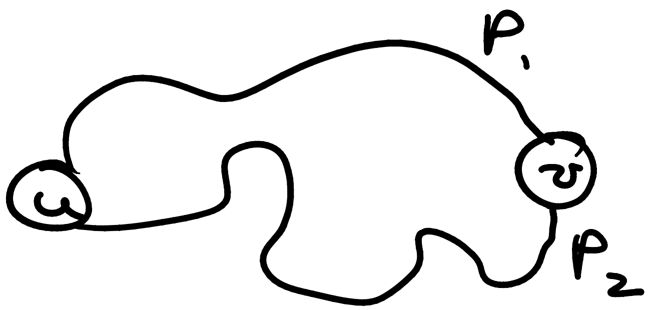
Thursday, February 23, 2023 4:01 PM

2-connectivity

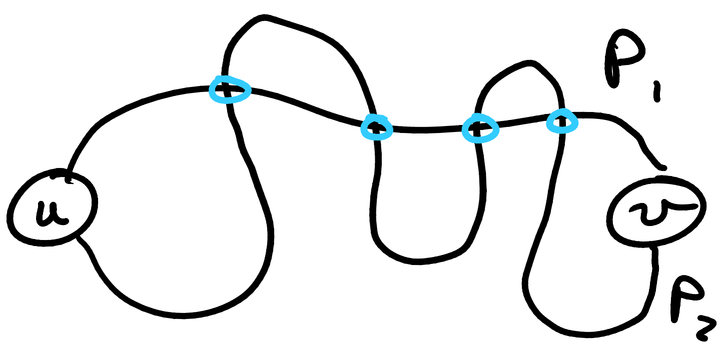
→ must remove 2 vertices to disconnect G

Internally-disjoint paths

→ paths between some u, v that share no internal vertex



Internally edge-disjoint paths



→ paths between some u, v that have no shared internal edges

(1932)

Whitney (1932)

$|V(G)| \geq 3$

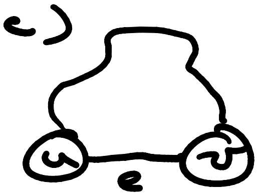
G is at least 2-connected

iff $\forall u,v \in V(G): \exists P_1 P_2$ u,v-idps

internally disjoints paths

($\Rightarrow$) Induction on $d(u,v)$ (distance)

Basis: $P(1) \to d(u,v) = 1$



$P_1 = e$

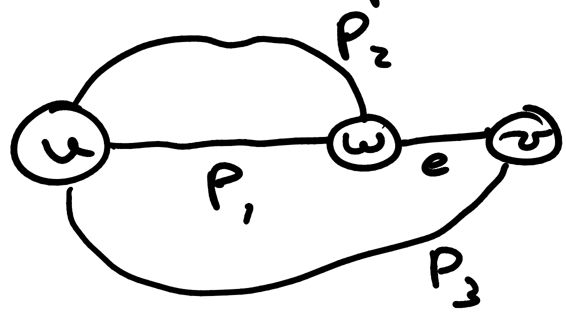
$P_2 =$ any path on $G - e$

recall $K(G) \leq K'(G)$

Consider G with u,v s.t. $d(u,v) = n$

$\to \exists$ at least on u,v-path

on this path, consider $w \in N(v)$



$d(u,w) = n - 1 = k$

I.H. $\to \exists P_1 P_2$ idps

P_3

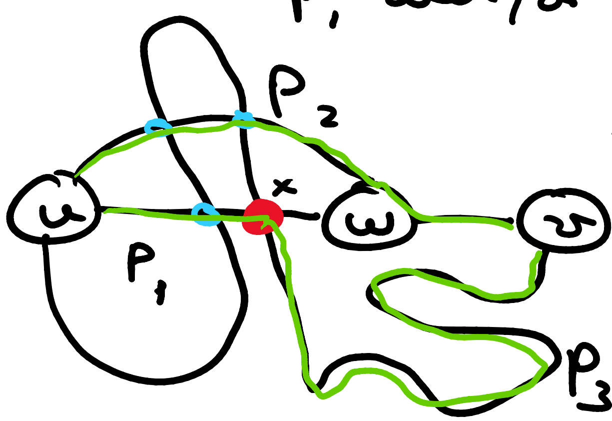
As G is 2-connected

$\exists\, P_3 = u,v$-path on $G - w$

Q: Is P_3 internally disjoint on P_1 or P_2?

Case 1: <u>Yes</u> $\rightarrow$ P_3 and $P_1 + (w,v)$ give us 2 idps

Case 2: <u>No</u> :($\rightarrow$ P_3 intersects P_1 and/or P_2 some # of times

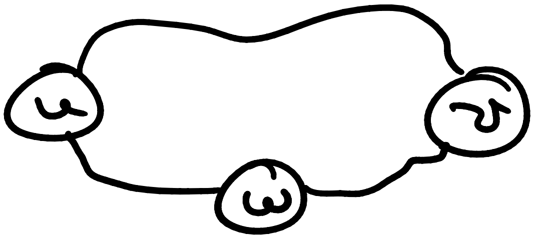


- define x as the last vertex on P_1 or P_2 that P_3 intersects (wlog say its on P_1)

Note: $P_2 + (w,v)$ and $P_1 \rightarrow x \rightarrow P_3 \rightarrow v$ give us our 2 idps ✓

($\Leftarrow$) ... $\in V(G)$

($\Leftarrow$) consider any $u, v \in V(G)$
and their 2 idps P_1, P_2



- consider any w on P_1 or P_2 and $G - w$

$\rightarrow$ There still exist at least 1 u,v-path

$\Rightarrow$ need to remove at least 2 vertices to disconnect G $\square$

2-connectivity equivalences

G is 2-connected $\Leftrightarrow$ $\exists P_1 P_2$ idps $\forall u, v$

$\Leftrightarrow$ G has no cut vertex

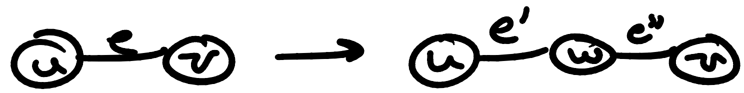
$\Leftrightarrow$ G has no cut edge

$\Leftrightarrow$ $\forall u, v \in V(G)$: $\exists C_n$ where $u, v \in C_n$

$\Leftrightarrow$ $\forall e, f \in E(G)$: $\exists C_n$ where $e, f \in C_n$

$\iff \forall e,f \in E(G): \exists C_n$ where $e,f \in C_n$

To show the last statement, we'll use the concept of subdivision

subdivision: 

Note: subdivision preserves 2-connectedness

So subdividing e,f to w, w' preserves our 2-connectedness and therefore 2 idps from w to w'

$\Rightarrow \exists C$ s.t. $w, w' \in C$

and equivalenty that cycle would have e,f in the original G $\square$

Ear decomposition

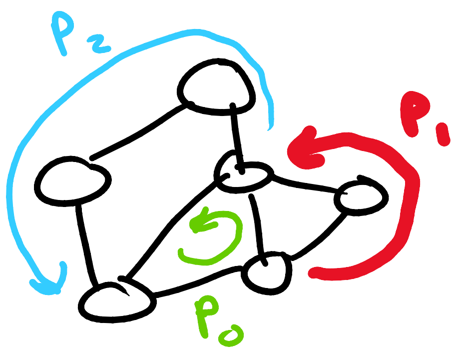
Open-ear decomposition is a decomposition of a 2-connected graph G s.t.

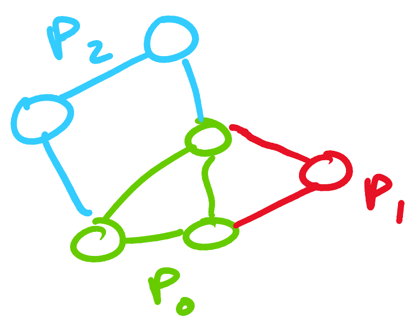
$D = P_0 P_1 P_2 \dots P_k$

P_0 = cycle

P_i = an open path whose endpoints exist on $P_0 \dots P_{i-1}$

$0 < i \le k$





Prove: G is 2-connected $\Longleftrightarrow$ G has an open ear decomp.

($\Longrightarrow$) PROOF BY ALGORITM

Select any arbitrary $u, v \in V(G)$

↪ $\exists\, C_n$ s.t. $u, v \in C_n$

$P_0 = C_n$

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while $\exists e \in E(G) : e \notin P_0 \ldots P_{i-1}$

consider any $f \in P_0 \ldots P_{i-1}$

$\hookrightarrow \exists C_m$ s.t. $e, f \in C_m$

$P_i = e$ follows C_m in both directions until reaching vertices in $P_0 \ldots P_{i-1}$

$\Rightarrow$ this builds our decomposition ✓

($\Leftarrow$) To show: prove adding an ear does not affect 2-connectedness of an open ear decomposition

First consider P_0, our cycle

$\hookrightarrow$ cycles are 2-connected

Next, consider some P_i

and some $v \in P_i$

and some $u \in P_0 \ldots P_{i-1}$

Q: can we find 2 ipds?

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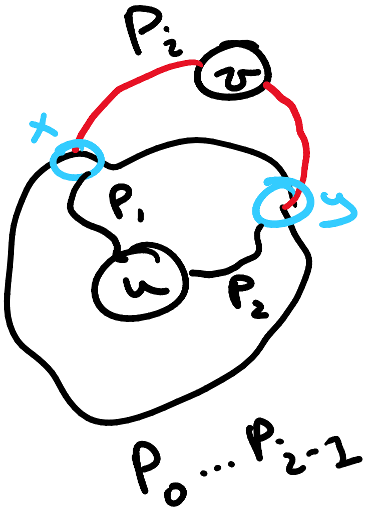
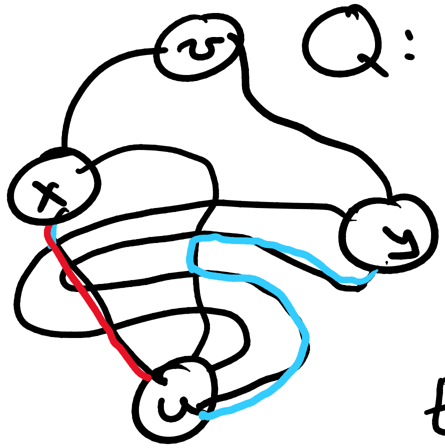


P_1 = v to x, then x to u along 1 u,x-idp.

P_2: v to y, then y to u along 1 u,y-idp



Q: are P_1 and P_2 internally disjoint?

→ Yes

Exercise 4 reader: Formalize the mess to the left

=> addition of an ear does not affect the 2-connectedness of our decomposition □

Closed ear decomposition

$D = P_0 \dots P_k$

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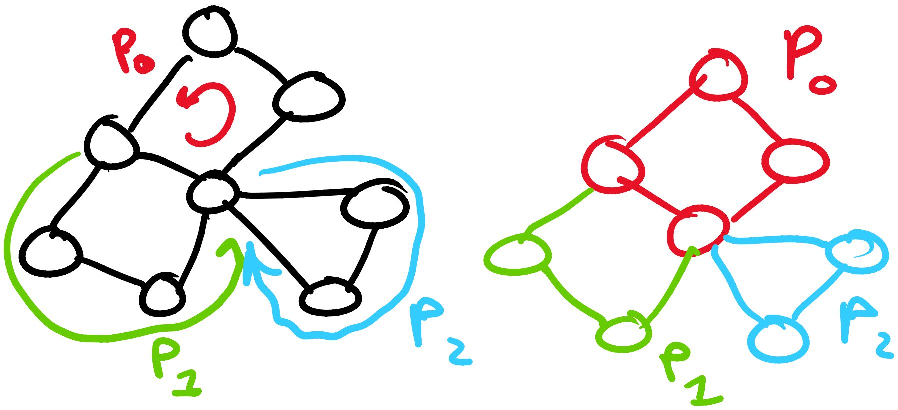
P_0 = cycle

P_i = open <u>or</u> closed path whose endpoints are on $P_0 \ldots P_{i-1}$



← closed ear

G is 2-edge-connected

$\iff$ G has no cut edge

$\iff$ G has a closed ear decomposition

proof same as above

$\iff$ $\forall u, v \in V(G) : \exists P_1 P_2$ internally edge-disjoint paths

proof same as whitney's

Biconnectivity

biconnectivity

→ a biconnected graph has no cut vertices

☆ K_1 and K_2 are biconnected

Block Decomposition of G

* <u>blocks</u> are maximal biconnected subgraphs
  aka <u>biconnected components</u>

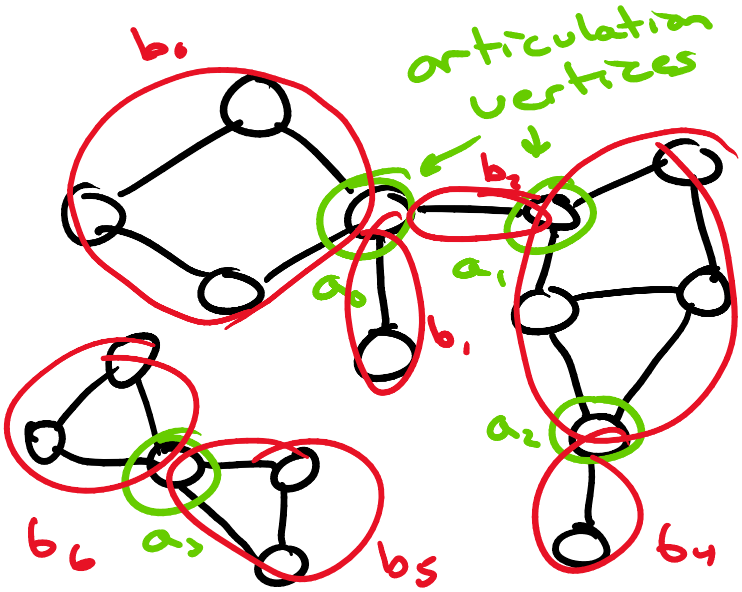
☆ <u>articulation</u> vertices are cut vertices

☆ <u>bridges</u> are cut edges

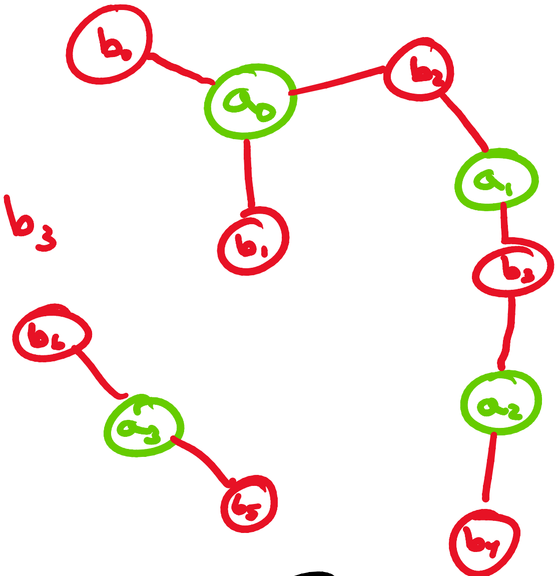
Using these, we can construct a <u>block-cutpoint graph</u> G_B

$V(G_B)$ = blocks & articulation vertices

$E(G_B)$ = articulation vertex membership in a block


articulation vertizes
b_0
b_1
b_2
b_3
b_4
b_5
b_6
a_0
a_1
a_2
a_3

G


b_0
a_0
b_1
b_2
a_1
b_3
a_2
b_4
b_6
a_3
b_5

G_B