Lecture 14 - k-connectivity

Thursday, March 16, 2023 12:50 PM

How 2 Prove 4 GT

Practice Practice Practice
- work through ones in class in book, etc.
- solve in multiple ways
  → You'll start seeing patterns
    * Bag o' tricks

Challenge #1
- We aren't doing strict mathematical proofs
- We're trying form logical connection
  Given property → shown prop.

Challenge #2
- What do we need to include?
  * Any time you make a statement

* Any time you make a statement about G's properties, it needs to be shown or given
* A lot of things seem obvious

↳ anything you use should not have "unproven" assumptions

E.g. you say "$\exists$ u,v-path"

Q: What do we need to know in order for the stament to hold?

A: G is connected

* Is the connected property explicitly given?

If not, we need to establish it

we wat to show $A \to D$

we need to show first

$A \to B$, $B \to C$, $C \to D$

Takeaway: include everything needed but in the simplest possible terms

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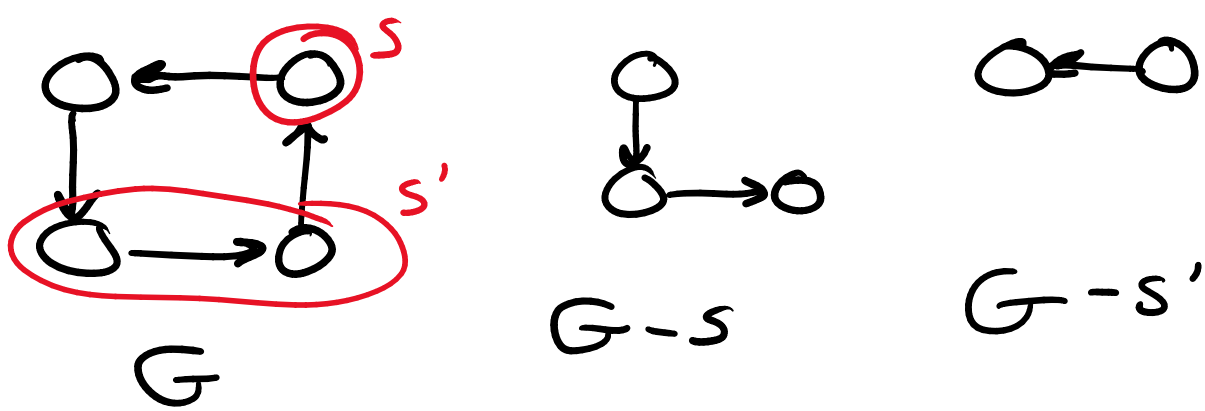
Digraph Connectivity

vertex cut – a set $S \subseteq V(G)$ s.t. $G-S$ is not strongly connected



$K(G)$ = connectivity of digraph G

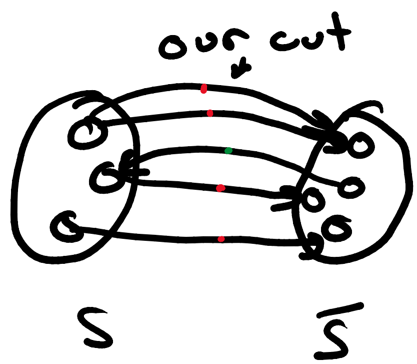
$= \min |S| \; \forall S \subseteq V(G)$ → aka size of a minimum cut

Edge-cut: a set $F \subseteq E(G)$

Edge-cut: a set $F \subseteq E(G)$ that separates $V(G)$ into two vertex sets $S, \bar{S}$

$$\bar{S} = V(G) - S$$

The size of this cut is the number of edges from $S \to \bar{S}$



cut of S: 4

cut of $\bar{S}$: 1

$K'(G)$ = edge-connectivity of G

$$= \min_{S \subseteq V(G)} \text{cut of } S$$

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k-connectivity

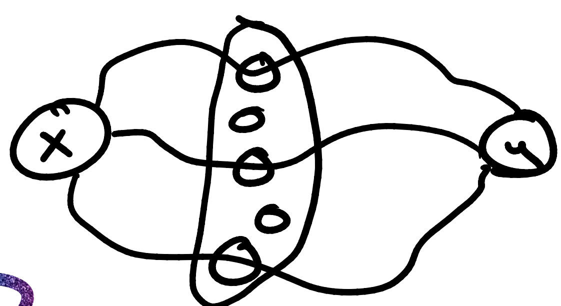
x, y-separator $\to$ a set $S \subseteq V(G)$ s.t. $G - S$ has no x, y-path

$K(G)$ = minimum x,y-separator over all $x,y \in V(G)$

$K(x,y)$ = connectivity of x,y
= minimum size of an x,y-separator

$\lambda(x,y)$ = minimum number of internally-disjoint x,y-paths

First note: every x,y-separator contains a vertex from each x,y-idp $\Rightarrow K(x,y) \geq \lambda(x,y)$



S

? ? ?

? Big Question

? ? ? ? ?

? ? ? ? ?

does $K(x,y) = \lambda(x,y)$?

Menger: yeah, it does
if $(x,y) \notin E(G)$

We shall use the **power** of strong induction to prove the above.

Induction on $|V(G)|$

Basis $P(2)$ $\quad x \quad y \quad \lambda(x,y) = 0$

$K(x,y) = 0$

$\lambda = K$ ✓

Assume we have some G with $|V(G)| = n$

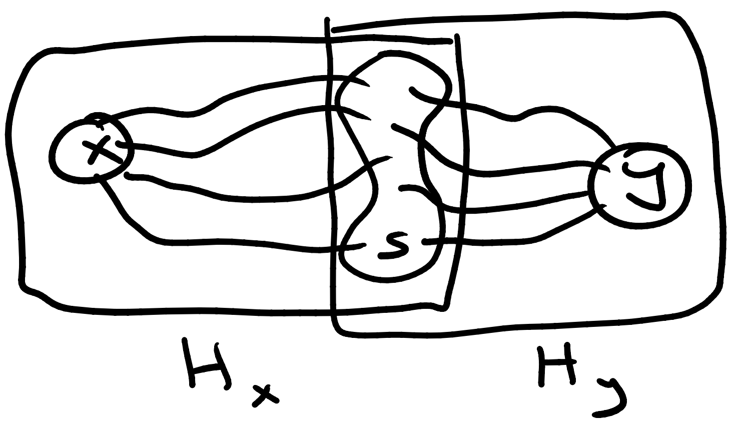
Also assume for some x, y, we have $K(x,y) = k = |S| \leftarrow$ min. sep.

have $K(x,y) = k = |S|$ ← min. sep.

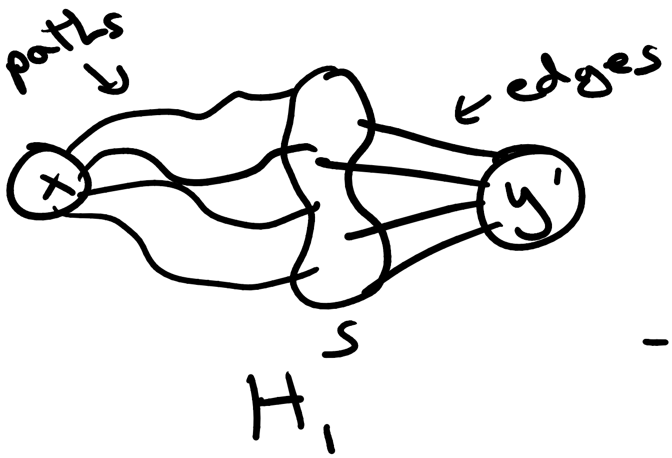
Our goal: construct k idps given our min separator S

Case 1: $\exists S$ s.t. $S \neq N(x)$, $S \neq N(y)$

- consider x,S-paths
- consider y,S-paths



- define graph $H_1 = H_x + y'$ and edges (s, y') $\forall s \in S$



I.H. on H_1 gives us k x,y'-idps

- define H_2 as $H_y + x'$ and edges (s, x') $\forall s \in S$

I.H. on H_2 gives us k

I.H. on H_2 gives us k x', y-idps

Taken together $\Rightarrow$ we have k x, S-idps and k S, y-idps, combined together we have our k x, y-idps ✓

Case 2: $S = N(x)$ or $S = N(y)$

2a) $\exists v \notin \{x\} \cup \{y\} \cup N(x) \cup N(y)$

Consider $G - v$

Note: v is not on a min cut

$\Rightarrow$ I.H. on $G-v$ gives us k x, y-idps and therefore k x, y-idps on original G ✓

2b) $\exists v \in N(x) \cap N(y)$

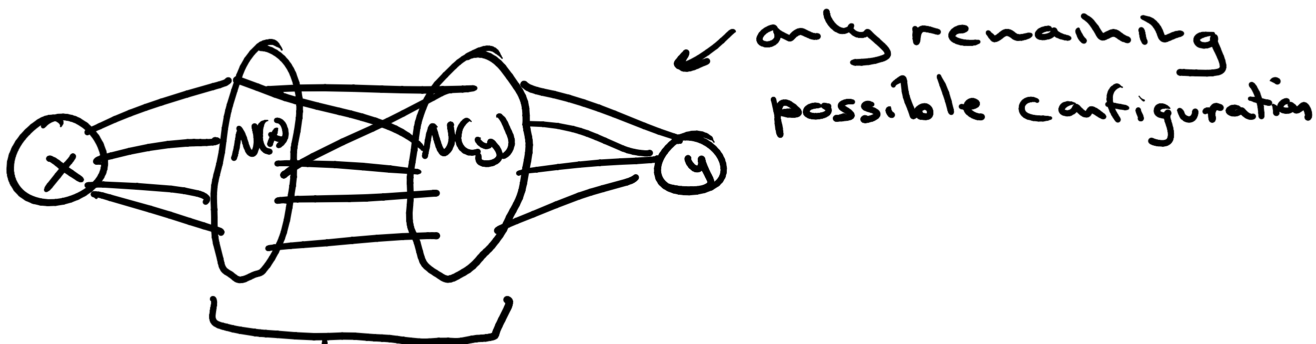
- Consider $G - v$

- Consider v ...

- I.H. on $G - v$ gives $(k-1)$ x,y-idps

$\Rightarrow$ we have our k^{th} idp on G with $(x,v)(v,y)$ ✓

2c) otherwise, both $N(x)$ and $N(y)$ are min. separators and there are no "external" vertices



we have a bipartite graph

Note: every x,y-path uses some edge in this bipartite graph

$\rightarrow$ we can construct idps by selected edges as a match

$|\text{max match}| = \text{max } \# \text{ of idps}$

$|\text{max match}| = \max \#$ of idps

Q: Can we guarantee k-idps or a match of size k on this bipartite graph $G_{N(x), N(y)}$?

We note that each of $N(x)$ and $N(y)$ are minimum covers on the bipartite graph

as $k = |N(y)| = |N(x)| = |\text{max match}|$ we have k matched edges

$\Rightarrow$ we can construct k-idps using this matched set $\square$

Same but different for k-edge-connectivity

$\kappa'(x,y) = x,y$-edge-connectivity
$= \text{minimum } x,y\text{-edge cut}$

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$\lambda'(x,y) = \text{max number of edge-disjoint paths}$

$\Rightarrow \kappa'(x,y) = \lambda'(x,y)$

G is k-connected if

$\forall x,y \in V(G): \kappa(x,y) \geq k$

$\lambda(x,y) \geq k$

G is k-edge connected if

$\forall x,y \in V(G): \kappa'(x,y) \geq k$

$\lambda'(x,y) \geq k$