14.1 Digraph Connectivity

We can extend the concepts and terminology of connectivity to directed graphs as well. A vertex cut or separating set in a digraph D is a set $S \subseteq V(G)$ such that D - S is not strongly connected. The **connectivity** $\kappa(D)$ is the minimum size of vertex set S such that D - S is not strongly connected or is a single vertex. If $k \leq \kappa(D)$, then D is k-connected. A digraph is k-edge-connected if every edge cut has at least k edges, where an edge cut separates V(D) into two sets S, \overline{S} such that the size of the edge cut is the number of directed edges (u, v) from $v \in S$ to $u \in \overline{S}$. The edge-connectivity $\kappa'(D)$ is the minimum size of an edge cut. If $k \leq \kappa'(D)$, then D is k-edge-connected.

As we have noted, 2-edge-connected graphs share similarities with strongly connected digraphs. We can show that adding a directed ear to a strong digraph produces a larger strongly connected digraph.

14.2 k-Connected Graphs

We can now further extend a few of the concepts we discussed with restriction to 2connected and 2-edge-connected to k-connected and k-edge-connected graphs. Given two vertices $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-separator if G - S has no x, y-path. We define $\kappa(x, y)$ as the minimum cardinality over all possible x, y-separators and $\lambda(x, y)$ as the maximum cardinality over all possible sets of internally disjoint x, ypaths. Since any x, y-separator must contain an internal vertex of every internally disjoint x, y-path, we have $\kappa(x, y) \geq \lambda(x, y)$.

What follows is a generalization of Whitney's Theorem. Menger's Theorem states that for two vertices $x, y \in V(G)$ and $(x, y) \notin E(G)$ the minimum size of an x, yseparator equals the maximum number of pairwise internally disjoint x, y-paths; i.e, $\kappa(x, y) = \lambda(x, y)$. A graph is therefore k-connected if for all $x, y \in V(G)$, $\lambda(x, y) \ge k$.

We have similar concepts and terminology for k-edge-connectivity. Given two vertices $x, y \in V(G)$, a set $F \subseteq E(G)$ is an x, y-disconnecting set if G - F has no x, y-path. We define $\kappa'(x, y)$ as the minimum cardinality over all possible x, y-disconnecting sets and $\lambda'(x, y)$ as the maximum cardinality over all possible sets of edge disjoint x, y-paths. A graph is k-edge-connected if for all $x, y \in V(G), \lambda'(x, y) \geq k$. Likewise, $\kappa'(x, y) = \lambda'(x, y)$.