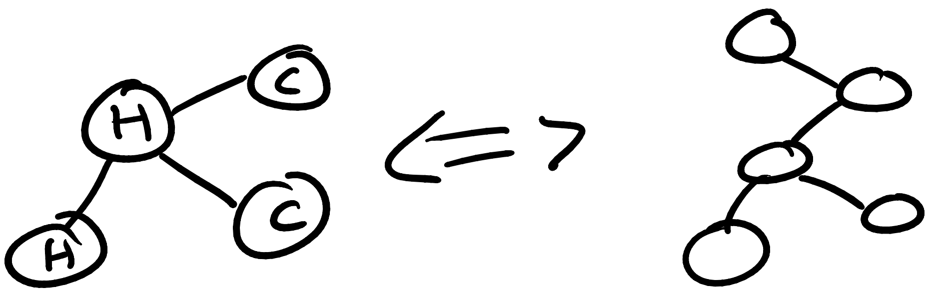
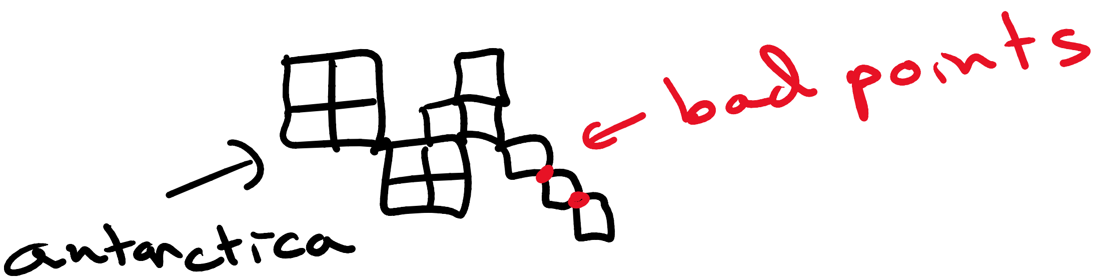
Applications

1. (sub) graph isomorphism

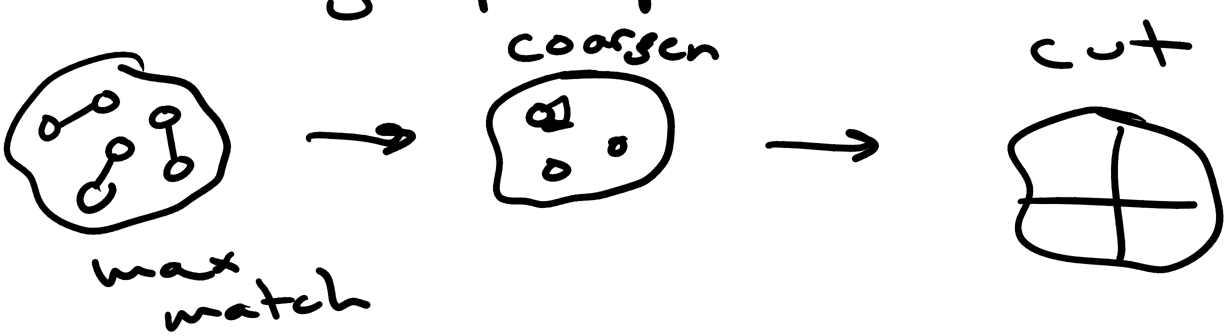


2. (bi) connectivity

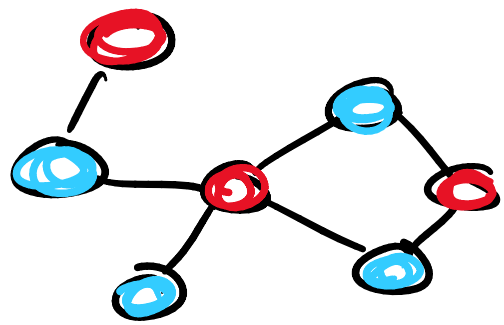


3. Matching

→ graph partitioning



④ Coloring



dependencies

Graph Coloring

<u>k-coloring</u> of G is a
labeling $f: V(G) \to S$, $k = |S|$

<u>proper coloring</u>: a k-coloring
of G s.t. no two
neighbors have the same color

<u>chromatic number</u> of $G \to \chi(G)$

$\chi(G)$ = the minimum k for
which G is properly k-colorable

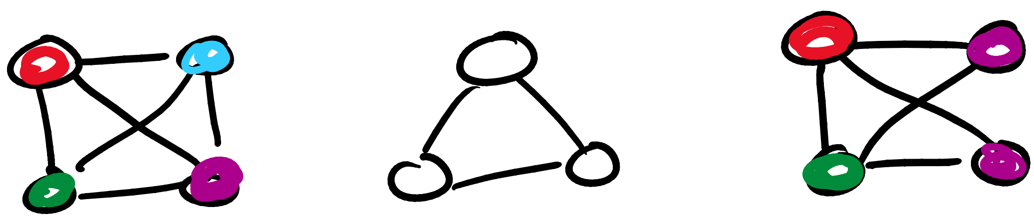
<u>optimal coloring</u> of G is
a proper k-coloring for $k = \chi(G)$

a proper k-coloring for $k=\chi(G)$

G is <u>color-critical</u> if for all subgraphs $H \subset G$, not $H \cong G$

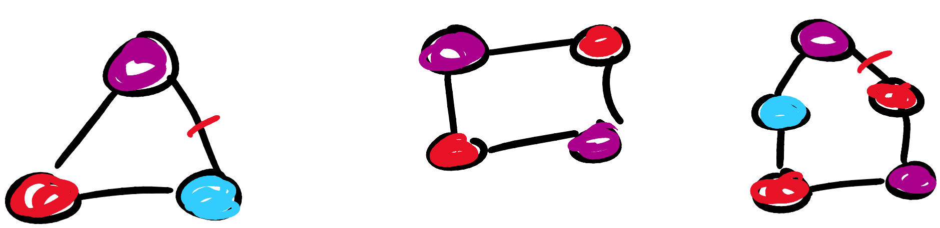
$$\chi(H) < \chi(G)$$

Note: all cliques are color-critical



K_4 $\qquad$ $K_4 - v$ $\qquad$ $K_4 - e$

Note 2: odd cycles are color-critical



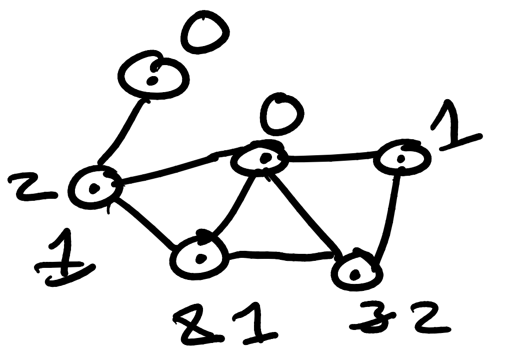
C_3 $\qquad$ C_4 $\qquad$ C_5

$$\chi(C_n : n = \text{odd}) = 3$$

$\chi(C_n : n = \text{even}) = 2$

Greedy Coloring Algorithm

all vertices have empty color
for all vertices in same order
    color vertex with "least"
    color that does not
    show up in N(vertex)



Note: greedy coloring is not optimal

→ Really, its quality depends on the processing order

Let's Talk Bounds

(on G's chromatic number)

In general for non-null G

$1 \le \chi(G) \le |V(G)|$

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If G is non-empty

$$2 \leq \chi(G)$$

If G is a tree

$$2 = \chi(G)$$

If G is bipartite

$$2 = \chi(G)$$

If G is a clique K_n

$$\chi(K_n) = n$$

If $\omega(G)$ = size of the largest clique in G aka the "clique number"

$$\chi(G) \geq \omega(G)$$

Consider our greedy algorithm

$$\chi(G) \le \Delta(G) + 1$$

Can we improve on this bound?

Brooks says: Yes we can

Brooks: $\chi(G) \le \Delta(G)$

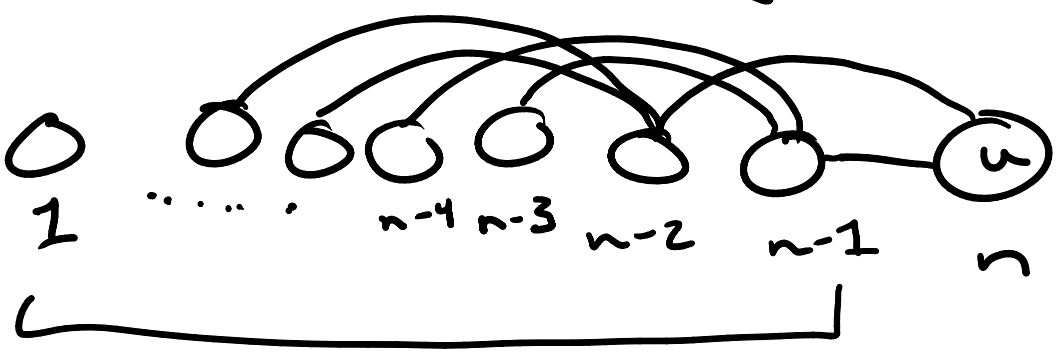
(except for cliques and odd cycles)

To prove: construct an ordering for greedy coloring s.t. we can guarantee each vertex has at most $k-1 = \Delta(G) - 1$ prior colored neighbors

Case 1: G is not k-regular

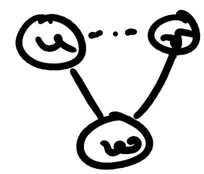
- select some $u \in V(G) : d(u) < \Delta(G)$
- grow a spanning tree from u
- apply order in reverse

→ every vertex is guaranteed at least one higher ordered neighbor



any $\Delta(G)$ vertex is in here

Case 2: G is k-regular

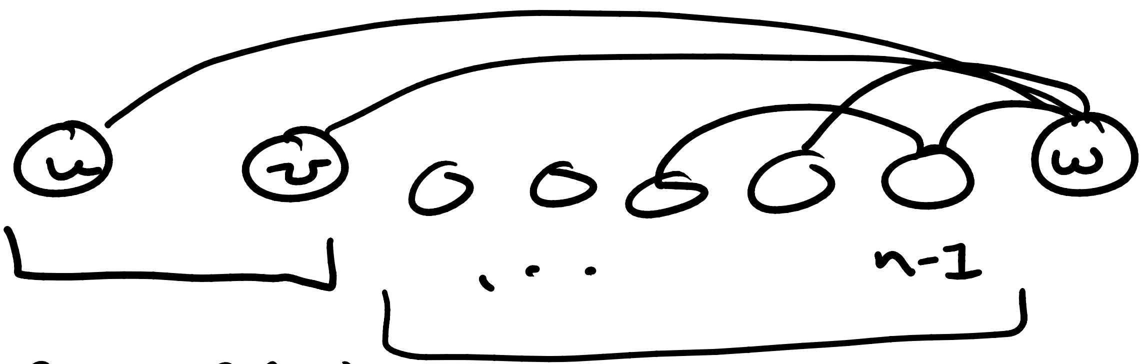


- consider $\{u, v\} \in N(w)$ s.t. $(u, v) \notin E(G)$

To construct our order

- u, v are listed first
- w is listed last

- we grow a spanning tree from w



$C(u) = C(v)$

has at least one higher-order neighbor

$\Rightarrow$ at most $\Delta(G) - 1$ colors show up in $N(x)$ for any x in this order $\square$

Q: how tight are these bounds???

A: not very

Consider a tree T

$\Delta(T) \to \infty$

$\chi(T) = 2$

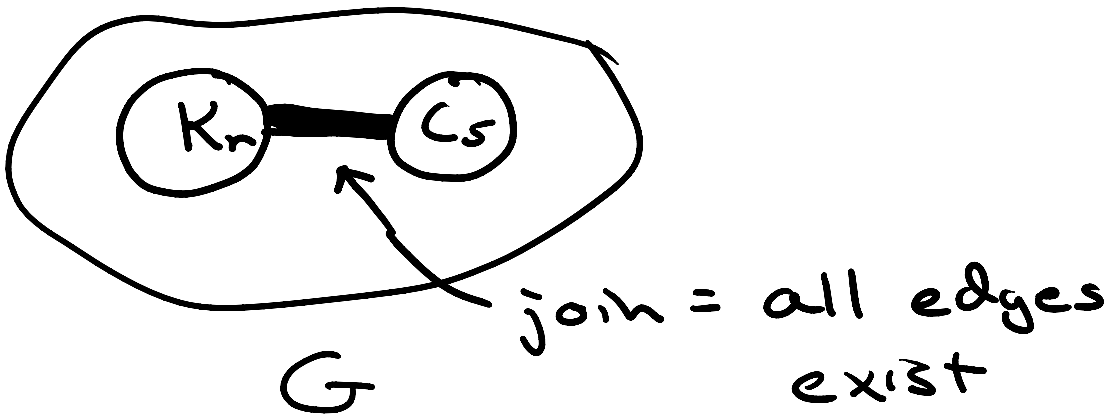
$\chi(T) <<<<<<<<<<<< \Delta(T)$

What about lower bounds?

$2 \leq \chi(G)$ if G is non-empty

$\omega(G) \leq \chi(G)$

First: consider some graph where we can guarantee inequality in the above



Note: we need $n+3$ to color the above G, while $\omega(G) = n+1$

$\omega(G) = n+1$

We saw that $\omega(G) \leq \chi(G)$ can be loose

Q: How loose?

A: ∞ ← infinitely

How can we show this?

→ via a construction that does not modify $\omega(G)$ but increases $\chi(G)$

aka Mycielski's Construction

→ Given a triangle-free graph with $\chi(G) = k$, we construct G' with $\chi(G') = k+1$ and $\omega(G) = \omega(G') = 2$

consider: $v_1 \ v_2 \ v_3 \ \dots \ v_n \in V(G)$

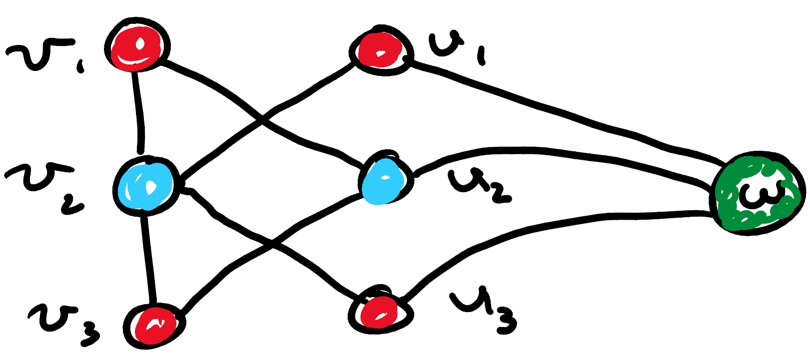
$\downarrow \quad \downarrow \quad \downarrow \qquad \downarrow$

create: $u_1 \ u_2 \ u_3 \ \dots \ u_n$

add edges between u_i and all $v_j \in N(v_i)$

create w

add edges from w to all u_i



G $\quad$ G'

$\chi(G) = 2$ $\quad$ $\chi(G') = 3$

$\omega(G) = 2$ $\quad$ $\omega(G') = 2$

Note: we don't create triangles

Note 2: coloring set of u vertices requires same # colors as for v vertices

Note 3: coloring vertex w requires a new separate a new separate color

We can iterate this construction infinite times

$$\Rightarrow \omega(G^{''''''''\cdots\times\infty}) <<<<<<<<<<<<< \chi(G^{''''''\cdots})$$

Takeaway: our bounds tell us very little in the general case

¯_(ツ)_/¯