Lecture 17 - k-chromatic graph properties

Friday, March 24, 2023 5:57 PM

Properties of a k-chromatic graph

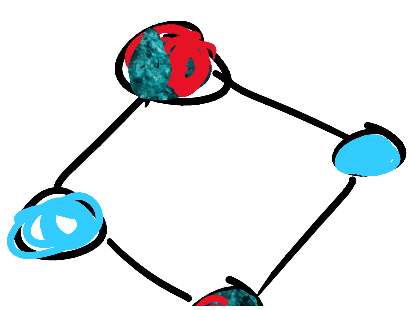
How small can a k-chromatic graph be?

Consider all possible color pairs

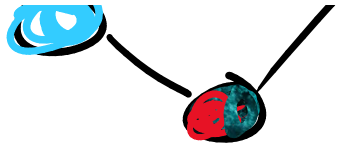
→ $\binom{k}{2}$ possible color combination on any given edge

→ this is the minimum number of edges for a k-chromatic graph

Why? Every combination must exist, as otherwise we could combine colors to get a k-1 coloring



possible color combos

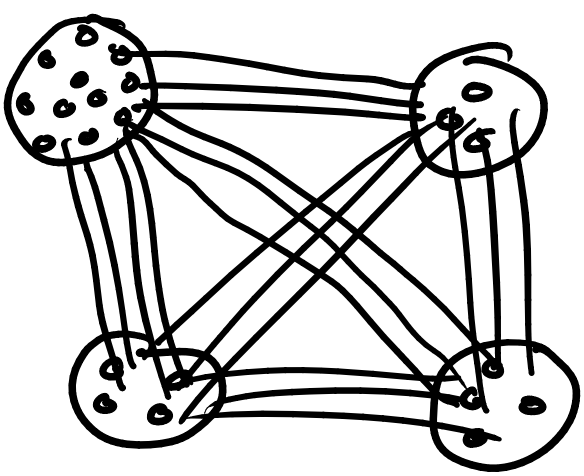


we can combine these to get a $k-1$ coloring

$\Rightarrow$ any k-chromatic graph must have at least $\binom{k}{2}$ edges

what about how BIG?

- consider a multi-partite graph (k-partite)



To maximize: make this k-partite graph complete

Can we further maxmize for a given k, $|V(G)|$?

→ set all partite set to be equal +/- 1 vertex

=> Turán graph

Q: Does it maximizes $|E(G)|$?

- consider an "unbalanced" complete multi-partite graph

- $\exists S_i S_j$ s.t. $|S_i| + 1 < |S_j|$

- move $v \in S_j$ to S_i

→ edges lost $= |S_i|$

→ edges gained $= |S_j| - 1$

as $|S_j| > |S_i| + 1$

we have a net gain

If we repeat this, we will

If we repeat this, we will eventually maximize $|E(G)|$

$\Rightarrow$ Turán graph is the largest possible k-chromatic graph on $|V(G)|$ vertices $\square$

Color-critical graphs

G is color-critical if

$\forall v \in V(G) \rightarrow \chi(G-v) < \chi(G)$

$\forall e \in V(G) \rightarrow \chi(G-e) < \chi(G)$

For color-critical graph G:

$\exists$ some k-coloring on G s.t. $\forall v \in V(G)$ the color $c(v)$ appears nowhere else and there is $k-1$ colors in $N(v)$

$\rightarrow$ consider $(k-1)$-coloring on $G-v$ & we add back v and

If we add back v and not all $k-1$ colors show up show up in $N(v)$

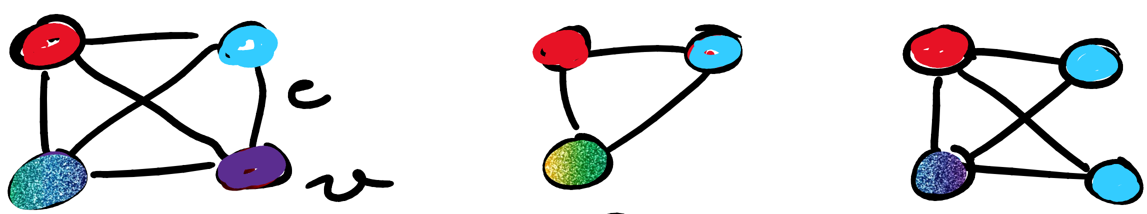
$\rightarrow$ we can assign to v a color not on $N(v)$

Contradiction

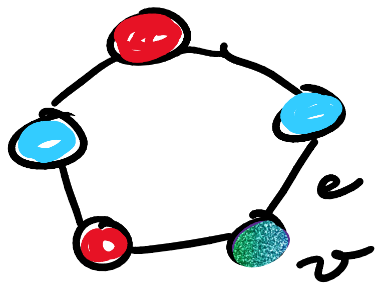
Similarly: $\forall e=(u,v) \in E(G)$

- Every proper $(k-1)$-coloring of $G-e$ gives $c(u)=c(v)$

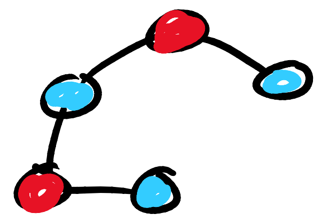
$\rightarrow$ If not, that would give a $(k-1)$-coloring on G



G $\qquad$ $G-v$ $\qquad$ $G-e$



G'



G' - e

Connectivity of a k-color-critical graph G

show: G is $(k-1)$-edge-connected

To do so, first show:

For G' s.t. $\chi(G') > k$, let $\{X, Y\}$ to be a partition of $V(G')$

If $G'[X]$ and $G'[Y]$ are both k-colorable $\rightarrow |[X,Y]| \geq k$

induced subgraph of G' on $X \subseteq V(G')$

consider $X_1, X_2 \ldots X_k$

consider $X_1, X_2 \ldots X_k$
and $Y_1, Y_2 \ldots Y_k$
as independent sets defined
by our assumed k-coloring

show: if $|[X, Y]| < k$, $\exists X_i Y_j$ that
we can combine to form a
k-coloring on G'

Assume $\underline{|[X, Y]| < k}$

construct H as a bigraph

$V(H)$ = each X_i, Y_j coarsened
to a single vertex

$E(H) = (X_i, Y_j)$ for all i, j
pairs where no edge
exist between $x \in X_i$
$y \in Y_j$ on G'

Note: H has more than $k(k-1)$
edges

[illegible]

edges

→ k^2 possible, but cut $< k$

Note x2: m vertices cover at most $m*k$ vertices in H

→ E(H) is not covered by only $(k-1)$ vertices

=> min cover $\geq k$
max match $\leq k$

min cover = max match = k

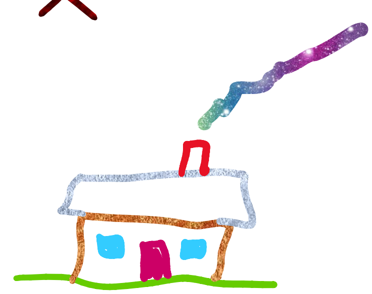
If we combine all matched sets into a single color

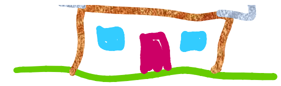
=> we get a k-coloring on G

Contradiction

=> $|[X, Y]| \geq k$

Bring it on home



Bring it on home 

→ show every k-color-critical graph is $(k-1)$-edge-connected

Consider k-color-critical G

$[X, Y]$ defines some min cut

→ $G[X]$ and $G[Y]$ are $(k-1)$-colorable

⇒ edge cut must be at least $(k-1)$ in size □

Minimum vertex coloring

Coloring G with $\chi(G)$ colors

Issue: NP-hard / complete

Takeaway: impossible to solve exactly in the general case

⇒ Heuristics and approximations are the name of the game

↳ usually based on greedy algorithm and vertex order

Breláz: go in order of which vertex currently has the most colors in its neighborhood

Logic: color the "most difficult" vertizes first

Optimal orderings for G.C. exist → show in HW5

From Q9: big takeaway is G.C. is Hugely dependent on processing order