Lecture 18 - Counting Colorings

Thursday, March 30, 2023 4:01 PM

$\chi(G, k)$ = # of way to color graph G with k colors

obviously

$\chi(G, k) = 0$ if $k < \chi(G)$

consider clique K_n and some k

$\rightarrow \chi(K_n, k)$

How can we determine this function?

$\rightarrow$ we can do this with some basic analytics

First, color any $v \in V(K_n)$ with one of k possible colors

Next, color second vertex with one of $k-1$ possible colors

... $k-2$

... $k-3$

... $k-3$

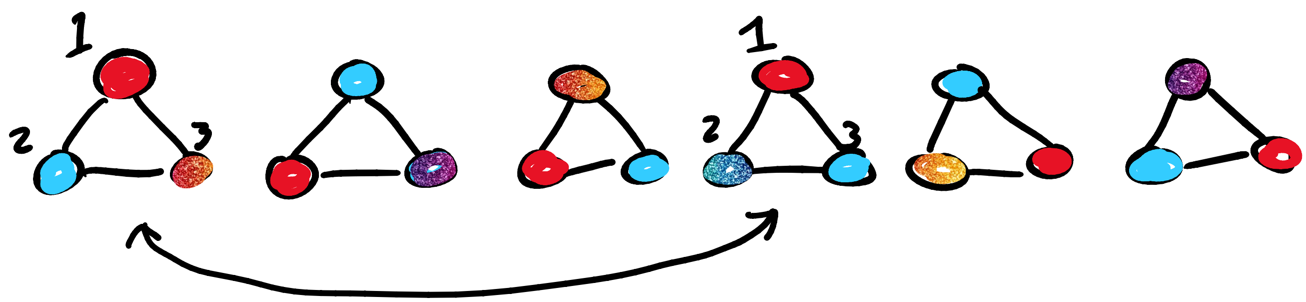
...

color final vertex with

$k-n+1$ possible colors

$$\chi(K_n, k) = k(k-1)(k-2)\cdots(k-n+1)$$

consider K_3 and $k = 3 = \{\text{red}, \text{blue}, \text{yellow}\}$



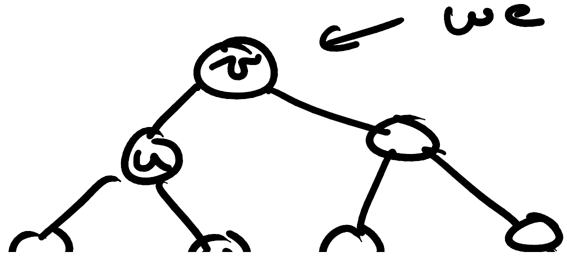
Let's talk trees

consider tree T and some

root $v \in V(T)$ and a

BFS from v

breadth-first-search

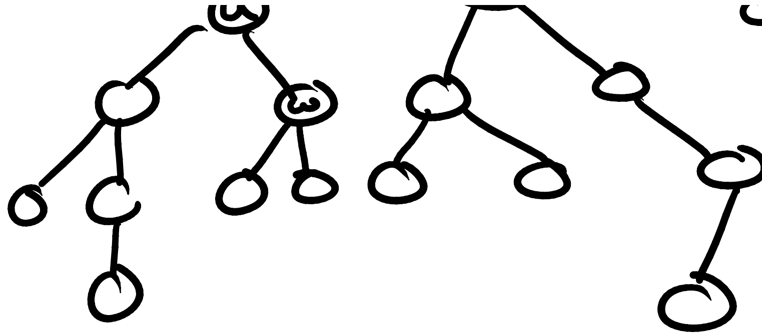


we can color

level by level

$c(v) = k$ possible

$c(u) = ?$



$c(v) = k$ possible

$c(w) = ?$

$(k-1)$ possible

→ any except for $c(v)$

$c(w)$ = any except for $c(w)$

What is $\chi(T, k)$?

Thus far, we've generally seen $\chi(G, k)$ to be a polynomial in k w.r.t. the structure of G

→ We call $\chi(G, k)$ the

Chromatic Polynomial of G

General form:

$$\chi(G, k) = \sum_{r=1}^{n} P_r(G)\, k_n$$

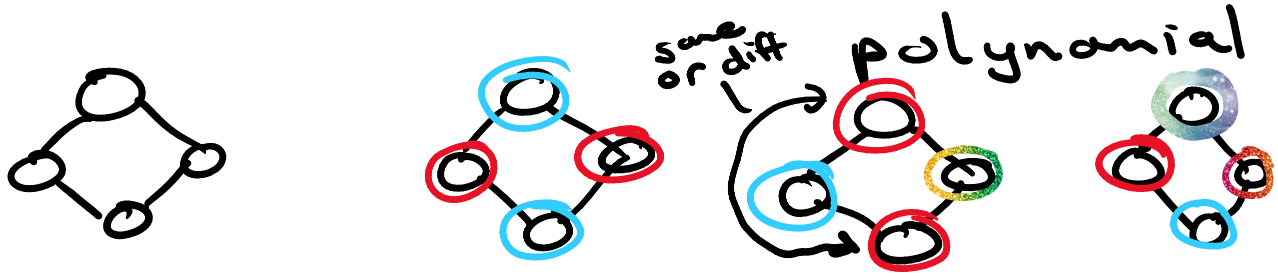
$P_r(G)$ = # of ways to partition G

$P_r(G)$ = # of ways to partition G into r independent sets (quite tough to calculate)

k_r = # of way to color r independent sets with k colors

$$k_r = k(k-1)(k-2)\cdots(k-r+1)$$

Consider C_4 and its chromatic polynomial



$P_1(C_4) = 0$ $\quad P_2(C_4) = 1$ $\quad P_3(C_4) = 2$ $\quad P_4(C_4) = 1$

$$\chi(C_4, k) = \sum_{r=1}^{k} P_r(C_4)\, k_r$$

$$= \underbrace{0}_{P_1} + \underbrace{1}_{P_2} k(k-1)$$

$$+ \underbrace{2}_{P_3} k(k-1)(k-2)$$

p_3

$$+ \underbrace{1}_{p_4} k(k-1)(k-2)(k-3)$$

$$\chi(C_4, k) = k(k-1) + 2k(k-1)(k-2) + k(k-1)(k-2)(k-3)$$

Q: what can we do with this?

A: determine C_4's chromatic #

$$\chi(C_4, k=0) = 0$$

$$\chi(C_4, k=1) = 0$$

$$\chi(C_4, k=2) = 2$$

nonzero, implies that

$$\chi(C_4) = k = 2$$

Q: can we derive $\chi(G, k)$ in a simpler way?

A: yes-ish

A: yes-ish

Fundamental reduction Theorem

$$\chi(G,k) = \chi(G-e,k) - \chi(G\cdot e,k)$$

$e = (u,v) \in E(G)$

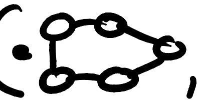
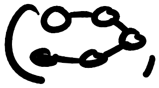
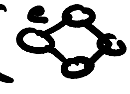
$\chi(G-e,k) =$ # of ways to color G where $c(u) = c(v)$ OR $c(u) \neq c(v)$

$\chi(G\cdot e,k) =$ # of ways to color G where $c(u) = c(v)$

Recall: $\chi(K_n,k) = k(k-1)(k-2)\ldots(k-n+1)$

$\chi(T,k) = k(k-1)^{n-1}$

Consider the above with C_5

$\chi($  $, k) = \chi($  $, k) - \chi($  $, k)$

tree

$= k(k-1)^4 - \chi($  $, k) + \chi($  $, k)$

tree clique

$$= k(k-1)^4 - k(k-1)^3 + k(k-1)(k-2)$$

Let's determine $\chi(C_5)$

$$\chi(C_5, k=1) = 0$$

$$\chi(C_5, k=2) = 2 \cdot 1^4 - 2 \cdot 1^3 = 0$$

$$\chi(C_5, k=3) = 3 \cdot 2^4 - 3 \cdot 2^3 + 3 \cdot 2 \cdot 1$$

$$= 48 - 24 + 6$$

$$= 30 \checkmark$$

Let's check ✓ it out

C_5 has 5 verts

As C_5 is color-critical, there exists a coloring where each of the 5 vertices has 3rd color

There are 3 color possible as 3rd

Other colors alternate on remaining vertices in 2 possible way

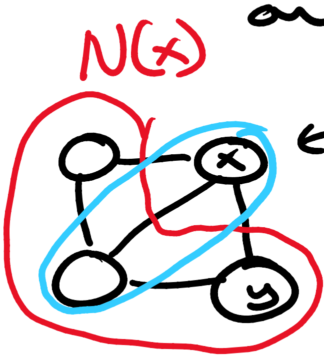
$\rightarrow 5 \cdot 3 \cdot 2 = 30$ ✓

Simplicial vertices

A <u>simplicial vertex</u> is a vertex v where $N(v) \cong K_n$

aka $N(v)$ is a clique

and $N(v) + v$ is a larger clique



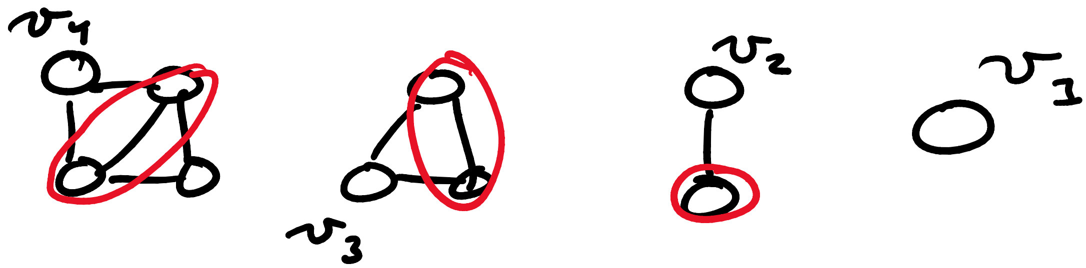
← not simplicial
$N(x) = P_3$

$N(y) = P_2 \cong K_2$ = clique

← yes simplicial

simplicial elimination ordering (SEO)

an ordering $\{v_n v_{n-1} \dots v_1\}$ of all $v \in V(G)$ for deletion, such that each v_i is simplicial in the remaining graph induced on $\{v_i v_{i-1} \dots v_1\}$



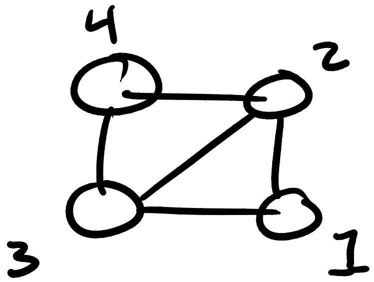
Why: we can construct a chromatic polynomial using this ordering

To get $\chi(G,k)$ using a SEO

→ add $v_1 v_2 \dots v_n$ to $G_i : G[\{v_1 \dots v_i\}]$

$$\chi(G,k) = \prod_{i=1}^{n} \left(k - d'(v_i)\right)$$

degree of v_i in G_i



$d'(v_1) = 0$

$d'(v_2) = 1$

$d'(v_3) = 2$

$d'(v_4) = 2$

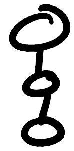
$\to \chi(G,k) = k(k-1)(k-2)^2$

Q10

P1: How to prove?

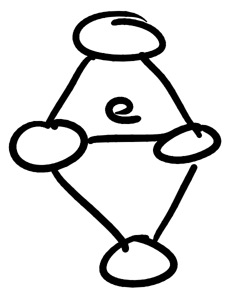
$\to$ Consider Fundamental Reduction Theorem

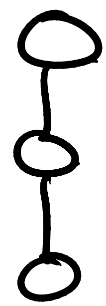
Note for coloring:

$\chi($  $, k) = \chi($  $, k)$

→ multi-edges are irrelevant for coloring problems

E.g.





G $\qquad$ $G \cdot e$

(effectively)