18.1 Counting Colorings

Although we can't directly determine if a general graph has a proper k-coloring, we are able to implicitly approach the problem through examining the number of possible colorings for fixed k values. Note that the computational complexities for both of these problem formulations are still exponential.

Given a Graph G and number of colors k, $\chi(G;k)$ is the number of proper colorings of G. If $\chi(G) < k$, then it is possible to create multiple colorings by altering the colors used on a proper coloring.

Consider clique K_n. The value of $\chi(K_n;k) = k(k-1)\cdots(k-n+1)$

If T is a tree with n vertices, then $\chi(T;k) = k(k-1)^{n-1}$.

We can recognize the fact that each k-coloring of G partitions G into k independent sets. Grouping the colorings according to this partition leads to a formula for $\chi(G;k)$ that is polynomial in k of degree n. $\chi(G;k)$ as a function of k is called the **chromatic polynomial**.

Let $k_{(r)} = k(k-1)\cdots(k-r+1)$. If $p_r(G)$ denotes the number of partitions of $V(G)$ into r nonempty independent sets, then $\chi(G;k) = \sum_{r=1}^{n} p_r(G)k_{(r)}$. Determining where this polynomial intercepts the x-axis would allow us determine the chromatic number of the graph we are computing it on.

However, directly computing the chromatic polynomial is generally infeasible. But, similar to how we counted spanning trees in G, there is a recurrence relation that we can use. If G is a simple graph and $e \in E(G)$, then the **Fundamental Reduction Theorem** states that $\chi(G;k) = \chi(G-e;k) - \chi(G \cdot e;k)$.

18.2 Chordal Graphs

A vertex is **simplicial** if its neighborhood in G is a clique. A **simplicial elimination ordering** is an ordering of vertices $\{v_n, \ldots, v_1\}$ for deletion such that each vertex v_i is simplicial in the remaining graph induced by $\{v_i, \ldots, v_1\}$.

We can create a chromatic polynomial of graph G with simplicial elimination ordering $\{v_n, \ldots, v_1\}$, by considering the graph induced by adding to its vertex set each vertex from G in order from v_1 to v_n. If we consider the possible colorings of the graph induced by e.g., $\{v_{i-1}, \ldots, v_1\}$, then the number of ways we can we can color to-be-added vertex v_i is $k - d'(v_i)$, where $d'(v_i)$ is the degree of vertex v_i in the induced graph G' with vertex set $V(G') = \{v_i, \ldots, v_1\}$. The chromatic polynomial would then just be the product of all $k - d'(v_i)$ terms for all possible $v_i : i = 1 \ldots n$.

A **chord** of a cycle C is an edge not in C but adjacent to two vertices in C. A **chordless cycle** in G is a cycle of at least length 4 that has no chord. A graph G is **chordal** if it is simple and has no chordless cycle.

A simple graph has a simplicial elimination ordering if and only if it is a chordal graph.

A graph G is **perfect** if $\chi(H) = \omega(H)$, where $\omega(H)$ is the size of the largest clique in H, for all induced subgraphs H of G. We can use the notion of simplicial elimination orderings to prove that chordal graphs are perfect.