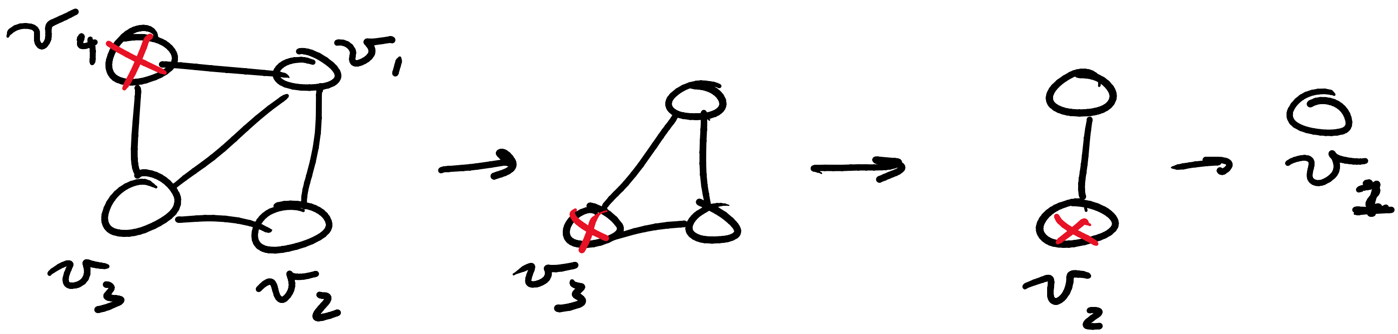
Lecture 19 - Graph Planarity

Sunday, April 2, 2023 1:28 PM

Recall: Simplicial elimination ordering (SEO)



working backwards

$$\chi(G,k) = \prod_{i=1}^{n} (k - d'(v_i))$$

$$\chi(G,k) = k(\underline{k-1})(\underline{k-2})(k-2)$$

$$\chi(G,0) = 0$$

$$\chi(G,1) = 0$$

$$\chi(G,2) = 2$$

$$\chi(G,3) = 3(2)(1)(1) = 6$$

Note: $\omega(G) = 3$

$\uparrow$

maximum clique size

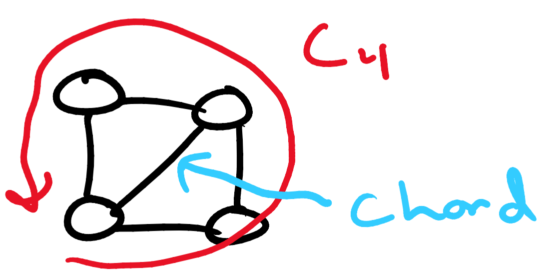
maximum clique size
aka clique number

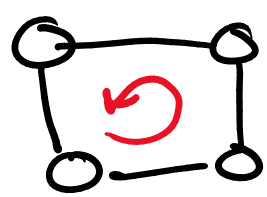
Q: What graphs have a SEO??

A: chordal graphs

chordal graph: simple graph that has no chordless cycle

Chord: an edge with endpoints on a cycle but is not part of that cycle





chordless

Chordless cycle: a cycle of at least 4 with no chords at least 4 with no chords

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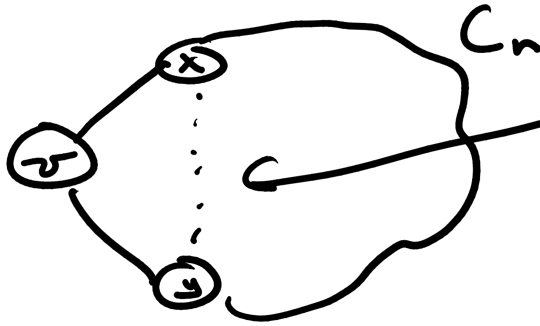
G has SEO $\iff$ G is chordal

($\Rightarrow$) Consider some $C_{n\geq 4} \subseteq G$

Consider the first $v \in C_n$ eliminated in our SEO



note: if $(x,y) \notin E$ v would not be simplicial

($\Leftarrow$)

Show: every chordal graph has a simplicial vertex farthest from some vertex x

Note: deleting a vertex can't introduce a chordless cycle

strong induction on $|V(G)|$

Basis $P(1)$: single vertex is simplicial

$P(n)$: we have G, $|V(G)| = n$

$\rightarrow$ consider $x \in V(G)$ (and $G - x$)

Case 1: $N(x) = \{V(G) - x\}$

$G - x$ is chordal

$\rightarrow$ I.H. on $G - x$

$\rightarrow$ any simplicial vertex in $G - x$ is simplicial in G

$SEO(G) = \{x\} + SEO(G - x)$

Case 2: $N(x) \neq \{V(G) - x\}$

- define T = vertices of maximum distance from x

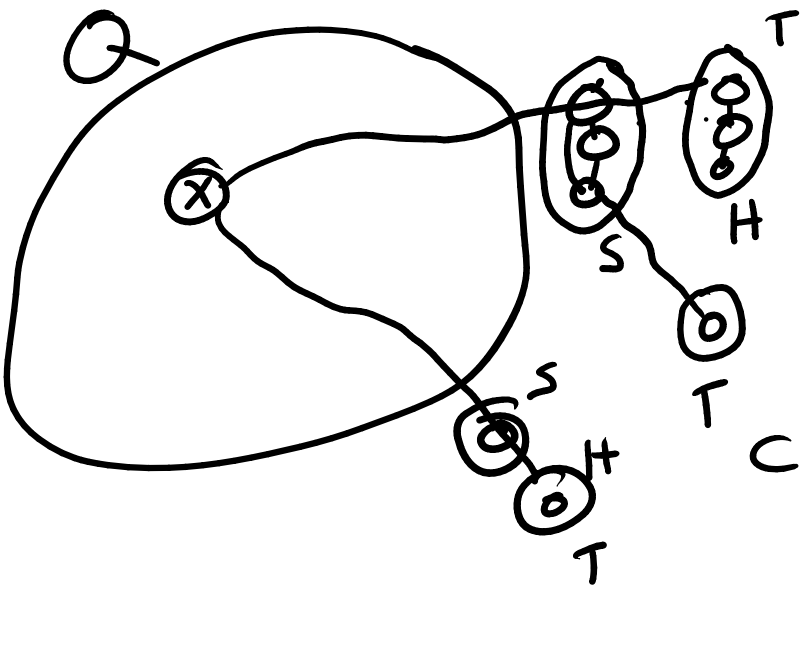
- define H = subgraph induced on $T \longrightarrow G[T]$

- define S = vertices in $G-T$ with neighbors in $V(H) \cup T$
- define Q = component of $G-S$ that contains x

Note: S must be cliques

→ has neighbors in Q and H for all $w \in S$

→ any cycle from $H \to Q \to H$ passing through $u,v \in S$ must have (u,v) as a chord



define $G' = G[S \cup V(H)]$

→ I.H. on G'

consider some $u \in S$

$\cup_T$ $\quad u \in S$

$\rightarrow \exists$ simplicial vertex in $H \rightarrow z$

$\hookrightarrow$ also simplicial in G

$\Rightarrow$ so we can construct a SEO using this vertex $\square$

Recall G is perfect if

$$\chi(H) = \omega(H) \quad \forall H \subseteq G$$

Show: chordal graphs are perfect

we know deleting a vertex cannot introduce a chordless cycle

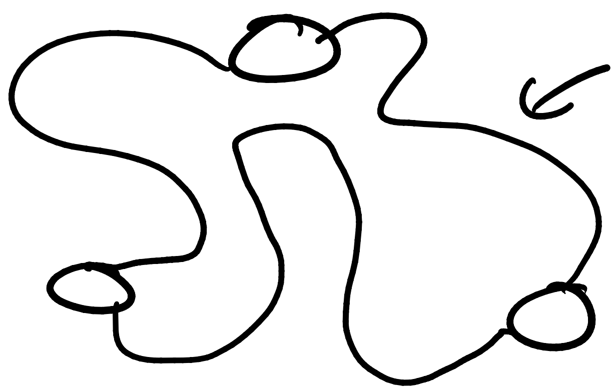
$\rightarrow$ show $\chi(G) = \omega(G)$ for all chordal graphs

We can then easily use a SEO to show the above a SEO to show the above

EXERCISE 4 Reader

Planarity and Drawing

Graph drawing: a mapping of vertices to points on the plane, and edges to some curves between those points



Graph Planarity: a graph is planar if it can be drawn without any crossings edge crossings

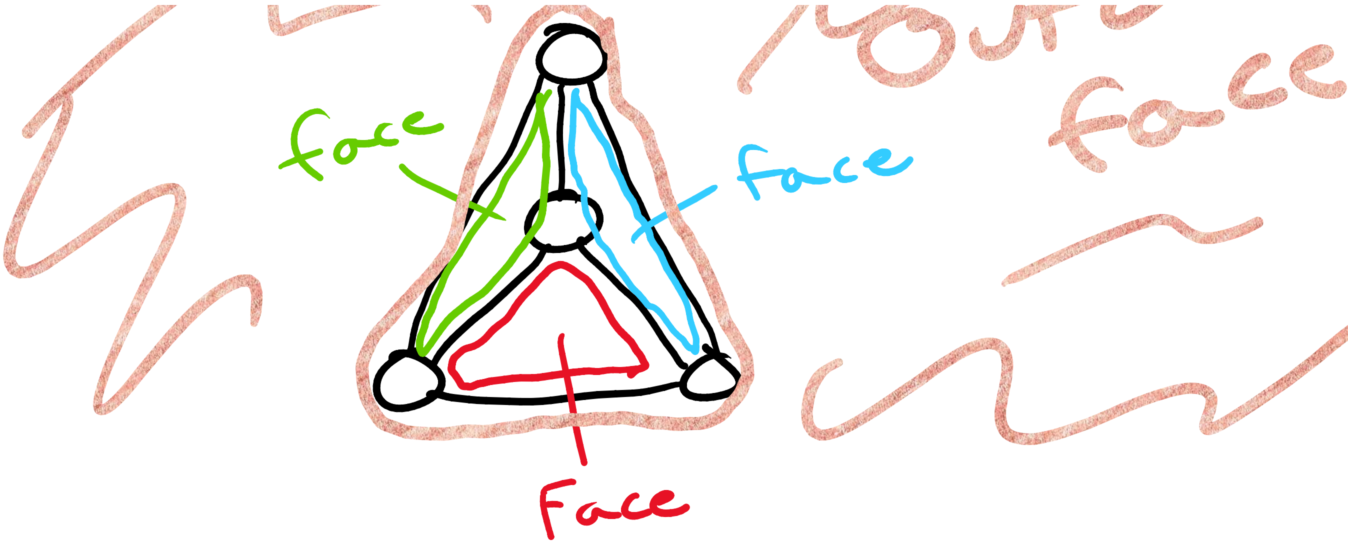
→ edge curve intersection

Couple more definitions:

planar embedding: graph drawing without edge crossings

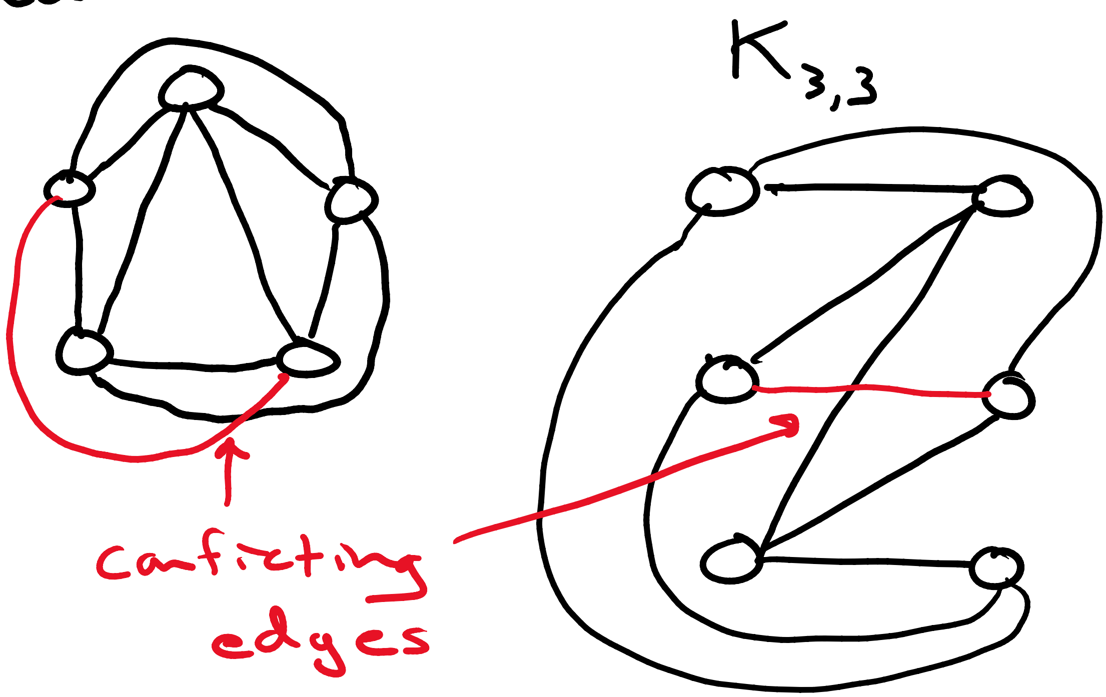
Face: maximal area in some embedding fully enclosed by edge curves

Outer Face: the external and unbounded face of the embedding



What about non-planar graphs?
aka: graphs with no possible planar embedding

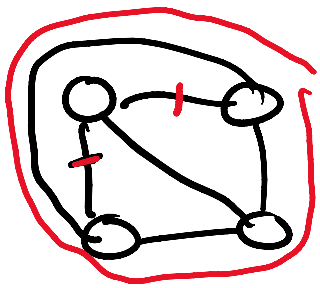
What about K_5?



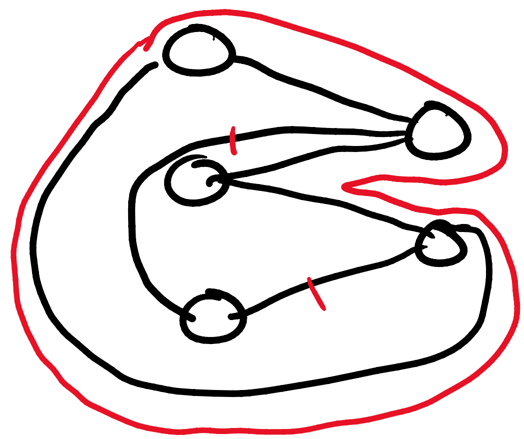
Outerplanar graph: a graph

Outerplanar graph: a graph with a planar embedding where all edges are on the outer face

K_4 is not outerplanar

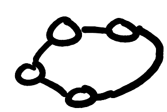


$K_{3,2}$ is not outerplanar



We'll formalize some of this later

Couple more examples



K_5 is not planar

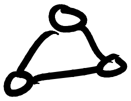
K_4 is planar but not outerplanar

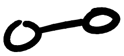
 K_3 is both

C_n is planar and outerplanar

trees?

$\rightarrow$ both

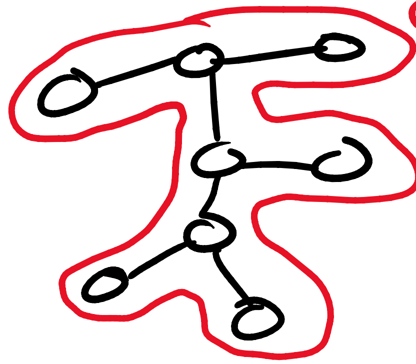
 K_3 is both

 K_2 is both

 K_1 is both

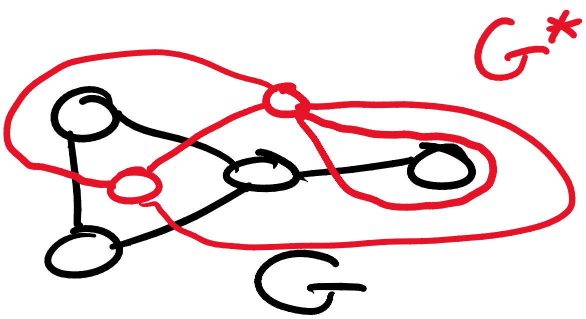
...ices.

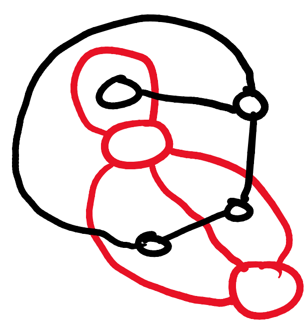
→ both



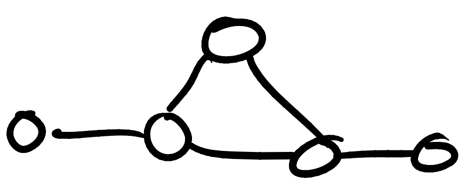
Dual Graphs

Dual graph G^* of embedding of G is a graph whose vertices are the faces of G and the edges are defined on the faces of G that share an edge

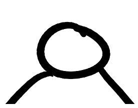


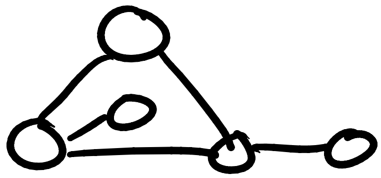


$(G^*)^* = G$



Note: G^* depends on a specific embedding





on a specific embedding of G

Note x2: the dual graph of a dual graph can be but is not always isomorphic to the original graph

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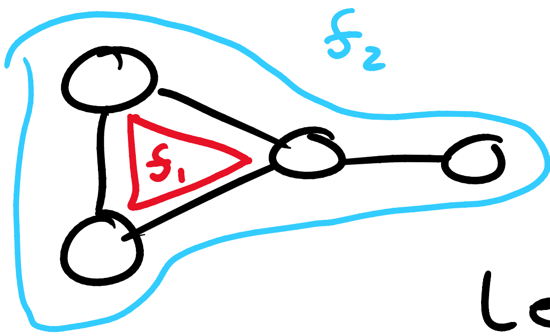
More on faces



G has two faces

Length of a face is the number of edges

$\ell(f_1) = 3$

$\ell(f_2) = 5$

Each edge contributes +2 to the sum of face lengths

$$\sum_i \ell(f_i) = 2|E(G)|$$

G is bipartite <=> all faces of G are even

<=> G^* is Eulerian

G bipartite => all faces are even

Note: all possible closed walks are even

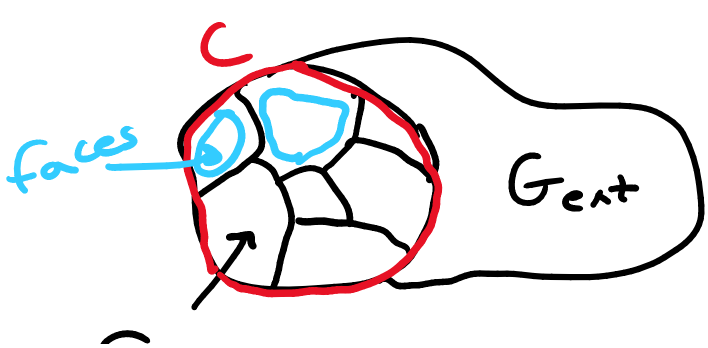
→ face lengths are defined by a closed walk

=> all faces are even

all faces even => G is bipartite

consider some cycle in G

all of G is either internal or external to that cycle in an embedding of G



G_{int}

Consider the internal portion of G

all faces are even

$\rightarrow \sum \ell(f_i) = \text{even}$

Note: each internal edge is counted twice

Note x2: each edge on C is counted once

1/0 1/0

Parity 1/0 $\Rightarrow$ the cycle of C must be even, regardless of the choice of C

1/0 1/0 1/0

$\Rightarrow G$ is bipartite

All faces even $\Leftrightarrow G^*$ is Eulerian

Note: vertex degrees of G^* are the lengths of faces of G

are the lengths of faces of G

all even => Eulerian

(<=) Same thing: Eulerian implies even degrees which implies even length faces in G $\square$