Lecture 20 - Kuratowski's Theorem

Sunday, April 2, 2023 1:26 PM

Euler's Formula

$$n - e + f = 2$$

n: $|V(G)|$

e: $|E(G)|$

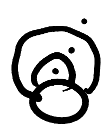
f: # faces in embedding of G ← connected and planar

We shall prove this using

POWER

of induction on n

Basis: $P(1) \rightarrow$  $n=1$, $f=1$, $e=0$ $\quad 1-0+1=2$ ✓

 $n=1$, $f=e+1$, $e=e$ $\quad 1-\cancel{e}+\cancel{e}+1=2$ ✓

Consider our $P(n)$ case

Note: there exists some edge that is not a selfloop edge that is not a selfloop

→ contract that edge to get P(k) case

I.H. on P(k)

↳ $n' - e' + f' = 2$

Bring it on back to P(n)

$n = n' + 1$

$e = e' + 1$

$f = f'$

plug n' chug

$n' - e' + f' = 2$

$(n-1) - (e-1) + f = 2$

$n - \cancel{1} - e + \cancel{1} + f = 2$

$n - e + f = 2$ ☑

QED

Let's use this formula

If G is a simple connected planar graph with $|V(G)| \geq 3$, then $e \leq 3n-6$

G is simple $\rightarrow \ell(f_i) \geq 3$

From our face length sum formula

$$2e = \sum \ell(f_i) \geq 3f$$

$$2e \geq 3f$$

consider $n - e + f = 2$

$$(e = n + f - 2) \cdot 3$$

$$3e = 3n + 3f - 6$$

$$3e \leq 3n + 2e - 6$$

$$\boxed{e \leq 3n - 6}$$

What if G is triangle-free?

$$\ell(f_i) \geq 4$$

plug n' chug

plug n' chug

$\boxed{e \leq 2n-4}$

Note: the above are

<u>Necessary</u>

but NOT sufficient conditions for planarity

maximal planar G: adding an edge to G makes it nonplanar

minimal nonplanar G: $\forall e \in E(G)$: $G-e$ is planar

triangulation: a planar embedding where all faces are of length 3

$\boxed{\text{maximal planar} \Leftrightarrow \text{triangulation}}$

note for the above

note for the above bound $e \le 3n-6$

Necessary conditions for planarity
(of a simple graph)

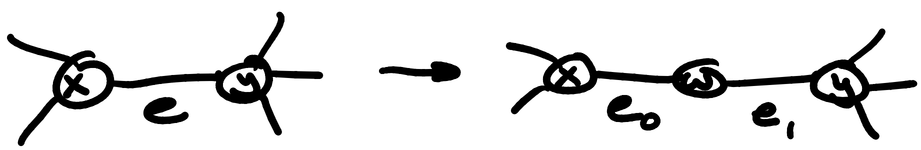
$e \le 3n-6$

$e \le 2n-4$ if G is triangle-free

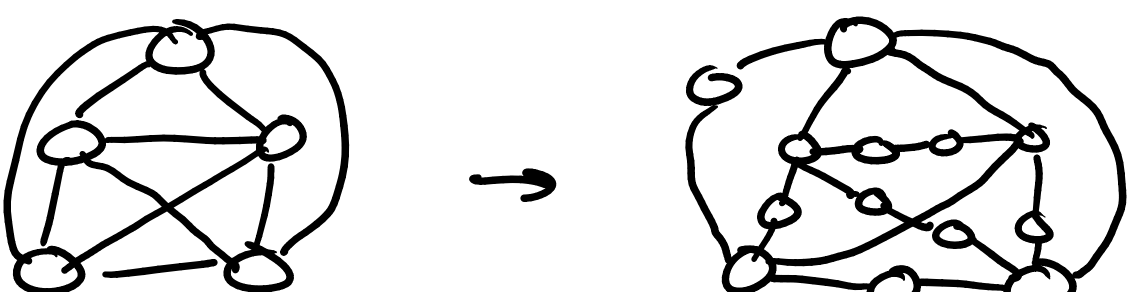
G has no K_5 or K_5 subdivision

G has no $K_{3,3}$ or $K_{3,3}$ subdivision

subdivided edge



subdivided subgraph



K_5 $\qquad\qquad$ K_5 subdivision

$\Rightarrow$ if H is nonplanar, then an H subdivision is nonplanar

K_5 or $K_{3,3}$ subdivisions

$\hookrightarrow$ Kuratowski subgraphs (K.S.)

Kuratowski's Theorem

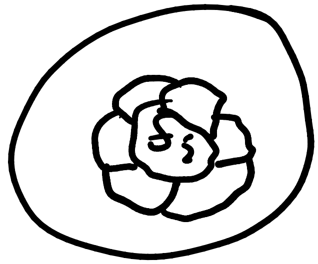
G is planar iff G has no K.S.

($\Rightarrow$) trivial

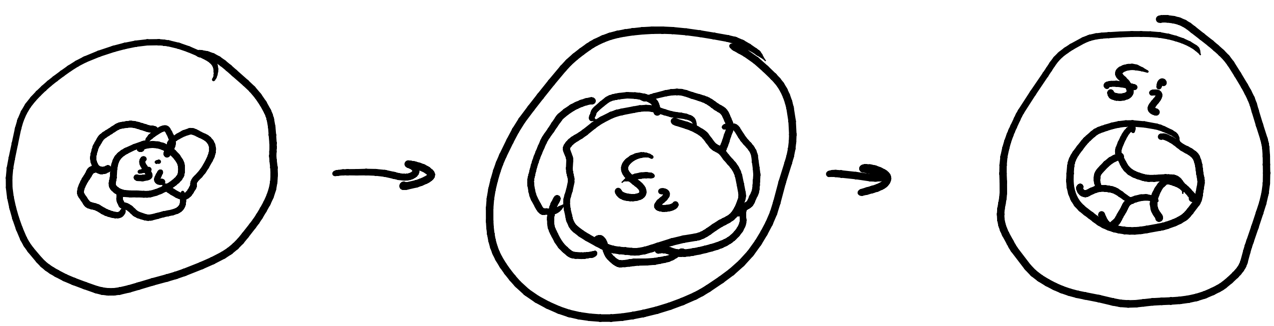
($\Leftarrow$) not so much

① For every face f_i of a planar embedding of G $\exists$ a planar embedding where f_i is embedding where f_i is the outer face

→ embed G on a sphere



→ expand f_i and return a projection of G bounded by f_i



(2) Every minimal nonplanar graph G is 2-connected

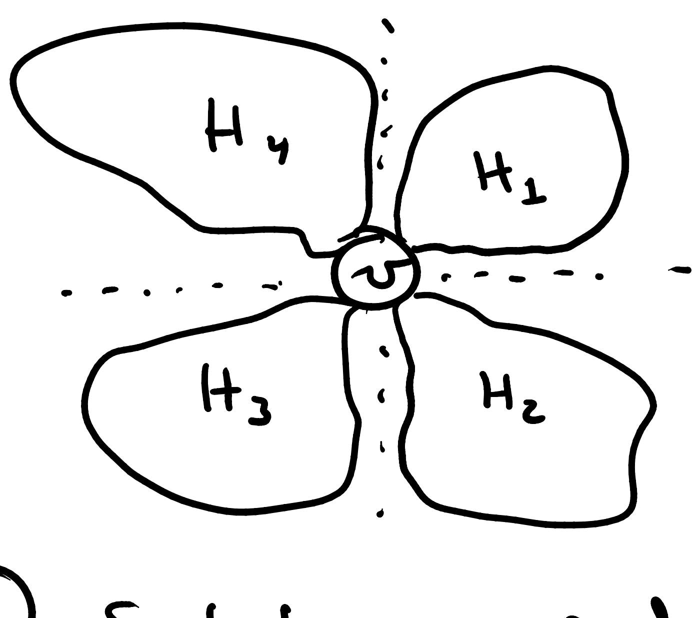
→ $\forall H \subset G$, H is planar

Assume $\exists v \in V(G) : G - v$ is disconnected

$G - v \to \underbrace{H_1 H_2 \dots H_k}_{\text{components}}$ (all H_i is planar)

We can create an embedding of G by "squeezing" all of H_i into $\frac{360}{k}$ degrees around v

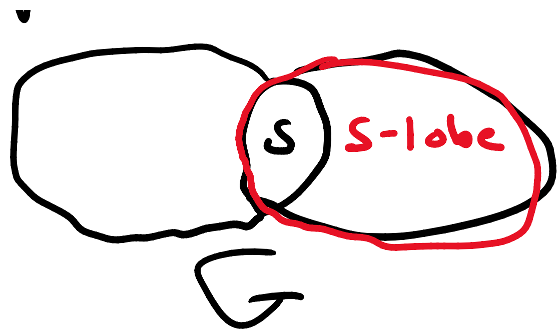
Note: From ① for all H_i there exists an embedding with v on the outer face



contradiction

$\Rightarrow$ G must be 2-connected

③ S-lobe: an induced subgraph of vertex set S and some component of $G - S$



Let $S = \{x, y\}$ be a separating set of some 2-connected G.

If G is nonplanar $\Rightarrow$ adding edge (x, y) to some S-lobe of G yields a nonplanar graph.

$\leftarrow$ component of $G-S$

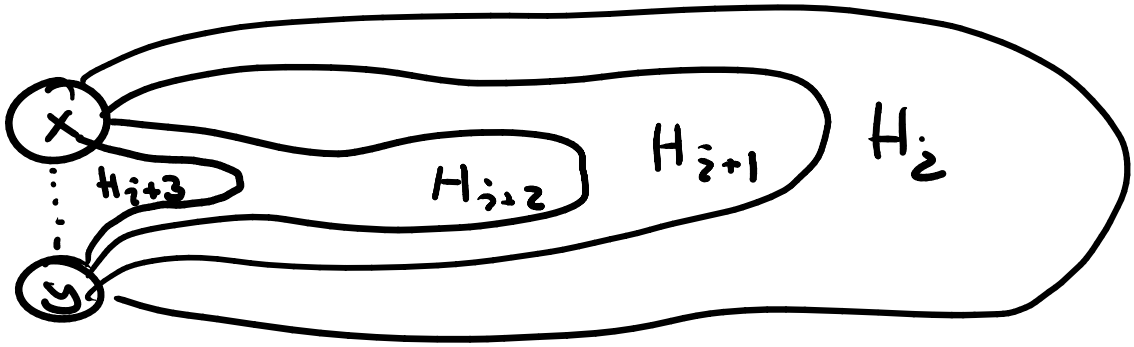
define $H_i = G_i \cup \{x, y\} \cup (x, y)$

S $\quad$ edge

If H_i is planar, from 1 it has an embedding where (x,y) is on the outer face

Assume all H_i are planar

$\rightarrow$ we can iteratively embed all $H_{i=2 \ldots k}$ into the face all $H_{i=2\ldots k}$ into the face of H_{i-1} containg (x, y)



contradiction

$\Rightarrow$ at least one H_i is nonplanar $\square$

(4) If G is a graph with the fewest edges among all nonplanar graphs without a K.S., the G must be 3-connected

Note: G doesn't exist, but if it did, it would need to be 3-connected

Why: restrict any possible counterexample to 3-connected

Why: restrict any possible counter-example to 3-connected graphs

Note x2: deleting an edge cannot create a K.S.

→ $G-e$ is planar and does not have a K.S.

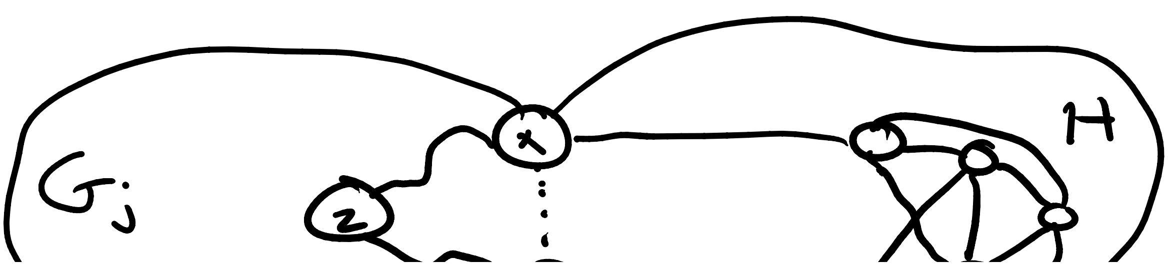
From ② G is 2-connected

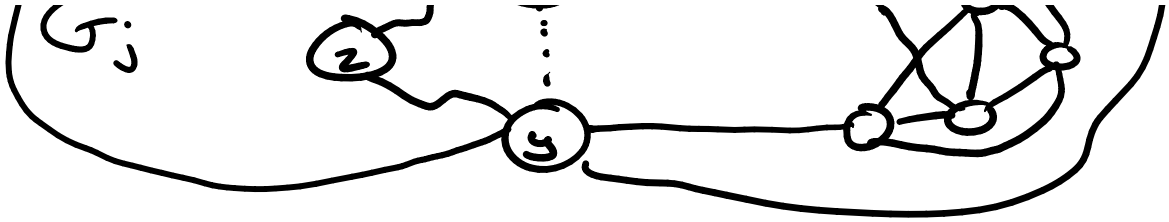
Assume $\exists S = \{x-y\}$, then some S-lobe of $G_2 + S$ is nonplanar from ③

define H as that S-lobe + (x, y)

From our minimality condition, H must have a K.S. $|E(H)| < |E(G)|$
and H is nonplanar

→ consider this configuration





However, since G is 2-connected, $\exists z \in G_j$, $G_i \neq G_j$ where we have 2 disjoint x,z and y,z-paths

$\Rightarrow$ we still have a K.S. when we consider these paths without edge (x,y)

✗ ✗ ✗
contradiction
✗ ✗ ✗

$\Rightarrow$ Any counter-example must be 3-connected $\square$

counter-example: nonplanar G with no K.S.

Next up...

Next up...

show all 3-connected graphs with no K.S. are planar

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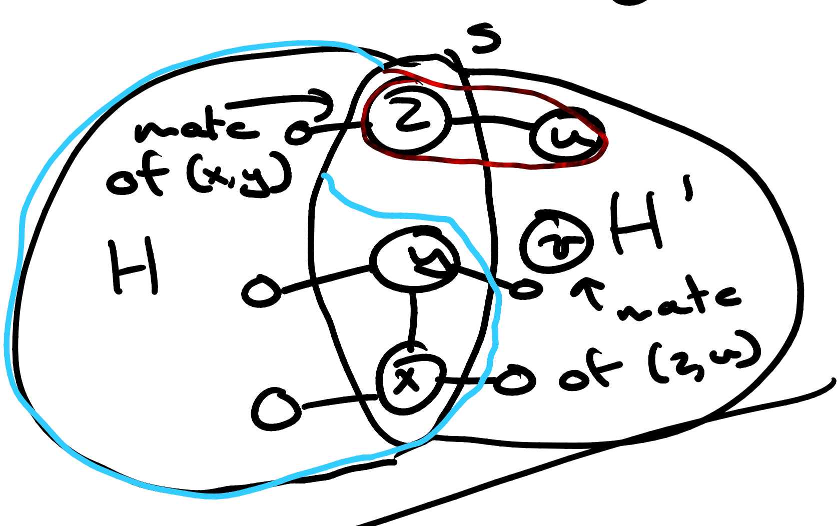
⑤ If G is 3-connected and $|V(G)| \geq 5$, $\exists e \in E(G)$ s.t. $G \cdot e$ is 3-connected

consider $e = (x, y) \in E(G)$ s.t. $G \cdot e$ is not 3-connected

$\exists S = \{x, y, z\}$



Assume $\exists$ no such edge s.t. $G \cdot e$ is 3-connected

→ all edges are within some separator with

same separator with
a 'mate' vertex

Select $S = \{x, y, z\}$ s.t.
$|V(H)|$ is maximum

Each of x, y, z have a neighbor
in each of H and H'
- consider $u \in N(z)$, $u \in V(H')$
- consider v, the mate
of edge (u, z)

$\rightarrow$ $G - \{z, u, v\}$ is disconnected

$V(H) \cup \{x, y\}$ is connected
and within a component
of $G - \{z, u, v\}$

Contradiction

of our selection of H

$\Rightarrow \exists e \in E(G)$ s.t. $G \circ e$ is

$\Rightarrow \exists e \in E(G)$ s.t. $G \cdot e$ is
3-connected $\square$

⑥ If G has no K.S.
$\Rightarrow G \cdot e$ has no K.S.

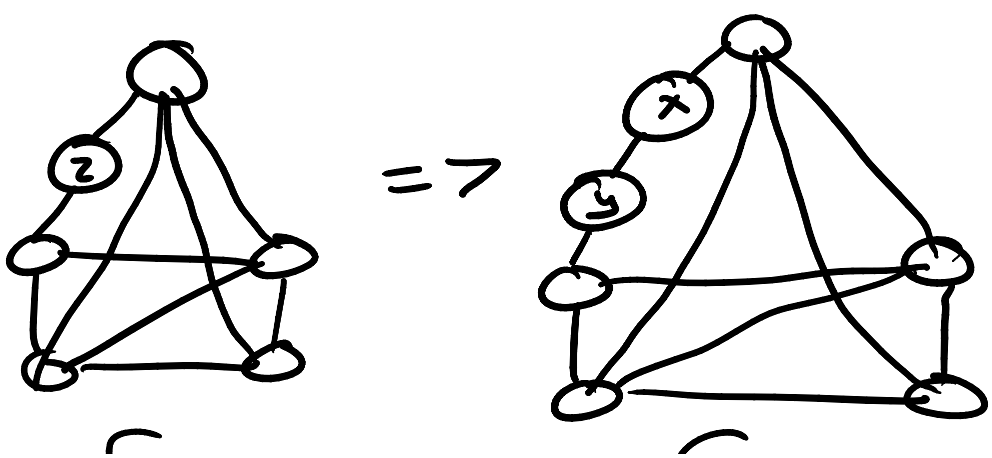
contrapositive

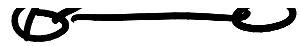
$G \cdot e$ has K.S. $\Rightarrow$ G has K.S.

define H as K.S. $\in G \cdot e$
$\leftarrow$ contracted edge
define $z \in V(G \cdot e)$, $z \leftarrow e = (x, y)$

Case 1: $z \notin H$
$\rightarrow$ trivially holds

Case 2: $d(z) < 3$ (degree in H)
$\rightarrow$ z is along a subdivided edge



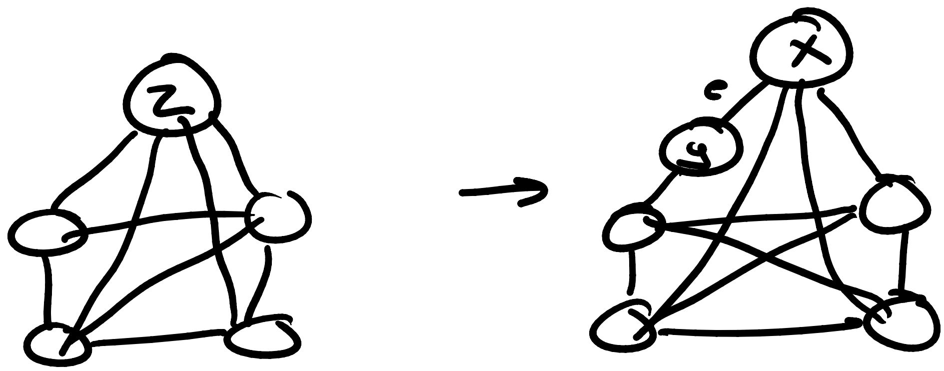


$G \circ e$

G

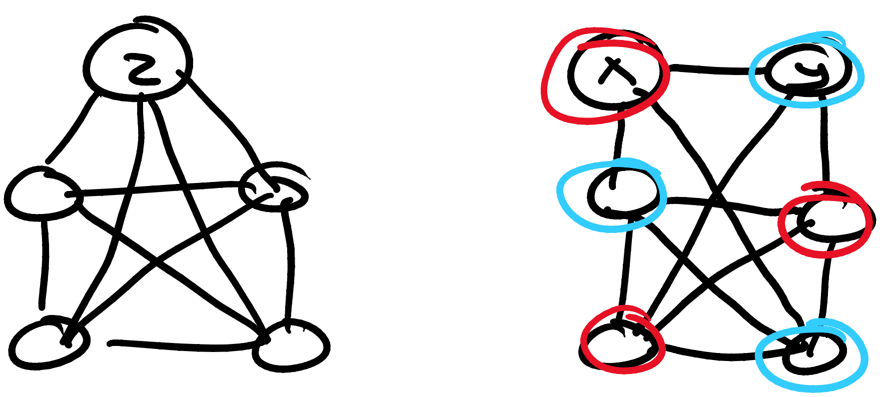
Case 3: $d(z) \geq 3$,
$d(x) \leq 2$ or $d(y) \leq 2$

$\rightarrow$ same thing, e is along a subdivided edge



Case 4: $d(z) \geq 3$
$d(x) \geq 3$ and $d(y) \geq 3$

$K_5 \rightarrow K_{3,3}$ is only way

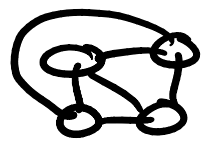


we have $K_{3,3}$ as a subgraph

Bring it on home

(7) If G is 3-connected with no K.S., G has an embedding on the plane

Induction on $|V(G)|$

Basis => K_4  is planar

Consider our $P(n)$ case

Note: $\exists e = (x, y)$ s.t.
$G \cdot e$ is 3-connected (5)

Note x2: if G has no K.S., then $G \cdot e$ has no K.S. (6)

Get $P(k)$ case via $P(n) \cdot e$

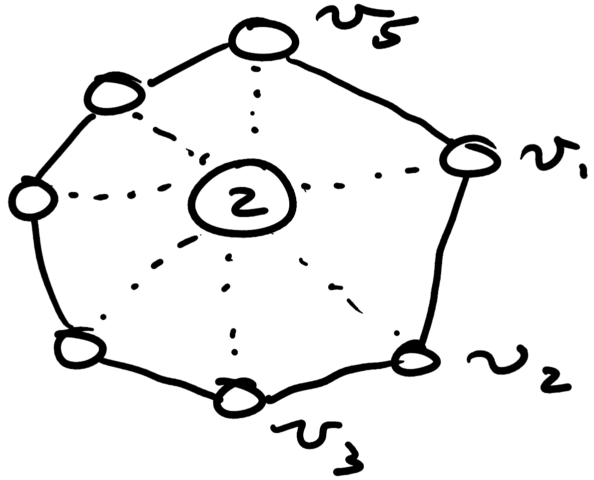
I.H. on $P(k)$

↳ $P(k)$ has an embedding on the plane

Bring it on back to $P(n)$

→ consider $z \leftarrow (x, y) = e$

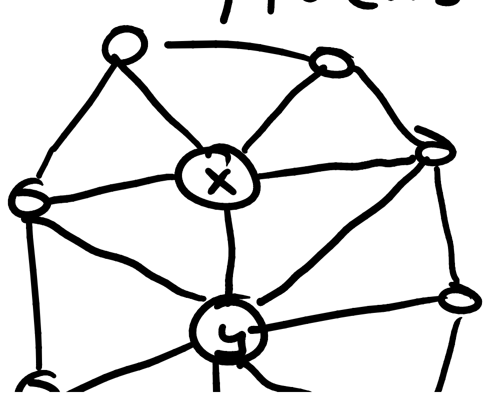
Note: all $N(z)$ form a face that contains z



- order all $N(z)$ as $v_1, v_2, v_3, \dots v_k$ around z
- consider what happens when $z \to (x, y)$

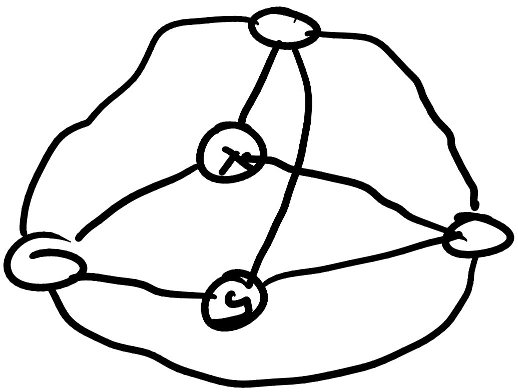
Case 1: $N(x)$ is some exclusive subset of $v_i \dots v_j$ and

$$|N(x) \cap N(y)| \leq 2$$



→ trivial to construct an embedding

Case 2: $|N(x) \cap N(y)| \geq 3$

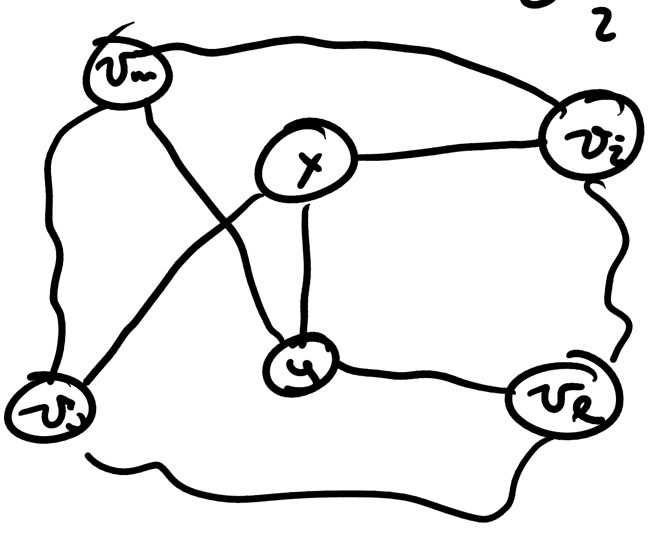


→ we have K_5 K.S.

Case 3: $N(x)$ alternates with $N(y)$ s.t. $v_i v_j \in N(x)$

$v_l v_m \in N(y)$

$v_i < v_l < v_j < v_m$



→ we have a $K_{3,3}$ K.S.

4 + 7 = Kuratowski's theorem

↘ no 3-connected

counter-example
must be 3-connected

no 3-connected
counter-example
exists

QED