Lecture 21 - 4/5 color theorems

Monday, April 10, 2023 1:04 PM

Story Time L

with Slota

aka "the realities of academic writing"

Combinatorial
Scientific
Computing
(solving graph prob. fast)

→ obvious measures of quality
speed, memory, scale

Q: In practice, how much does "being the best" count for?
papers, awards, etc.

→ Very little, unfortunately

Reason: Reviewers are going to

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Basically: we don't deserve to take 2x, 3x, 4x as much time for our work

The obvious solution

→ make everything as obvious as possible: "idiot proof"

Not that reviewers are idiots (very very far from)

Idiot Proof:

- Minimizing complexity of presentation
- Sticking to common formatting, basic practices, terminology
- Making it look "good" in

- Making it look "good", in terms of formatting, visually, sentence structure, etc.
  → nice to look at, easy to read

Over-complexity, length, complex wording or logic where simple ideas suffice
  → not easy to read

Takeaway → as many famous writers and speakers have stated: "I lacked the time to write a shorter letter"

Now: Graph Theory

The TA is grading 1000s

The TA is grading 1000s
and 1000s of proofs

→ think of the practical
consequences

→ think of and apply
all of the above

For those not currently here?

Is it fair to lose points
for a logically correct proof?

→ Debateable, but irrelevant

* We don't live in a
magical fairy land that
ignores basic practicalities

* We live in a practical
existence constrained by

existence constrained by
time, fatigue, and imperfections

People who find success tend to recognize and exploit this basic fact.

→ Recognize what you need to do, not what you wish you should need to do, and do it.

Show: all planar graphs are a subgraph of same triangulation

→ Kuratowski assumed this for final subproof

→ 4/5-color theorems centrally use this

Actually, pretty easy to show

consider some embedding of planar graph $G \to G \subseteq G'$

G' is a triangulation

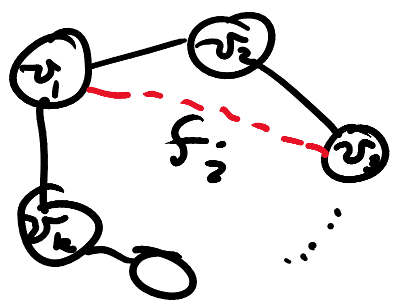
construct G':

$G' = G$

while G' is not a triangulation

$\exists f_i$ face of G', $|f_i| > 3$

$f_i = \{v_1 v_2 \ldots v_k\}$



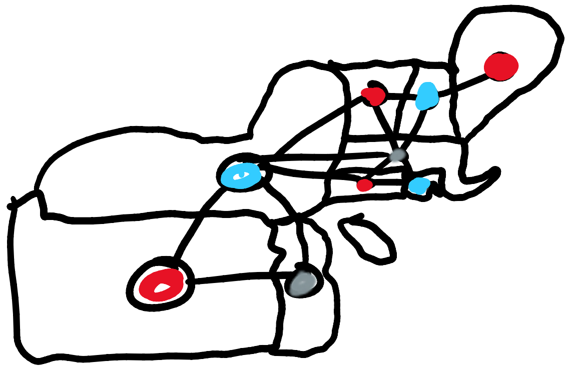
$G' = G' + (v_1, v_3)$

And we're done $\square$

Map Coloring

$\hookrightarrow$ vertex coloring of a planar graph

↳ vertex coloring of a planar graph



Q: How many colors do we need?

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sub Q: Can we bound $\chi(G) \leq 5$ when G is planar

5-color theorem: yes we can

Approach: find a minimum counter-example

Note: such a counter-example must have some v: $d(v) \leq 5$

↳ consider $m \leq 3n-6$

$\hookrightarrow$ consider $m \leq 3n-6$

and $\sum d(v) = 2m$

if all $d(v) = 6$

$2m = \underbrace{6n}$

$m \leq 3n-6$

$2m \leq 6n-12$

$6n \leq 6n-12$ X

$\Rightarrow \exists v : d(v) \leq 5$ in a planar graph

5-color theorem: $\chi(G) \leq 5$ if G is planar

Induction on $|V(G)|$

Basis $P(\leq 5) \rightarrow$ trivial to color

$P(n)$: consider planar G

where $|V(G)| = n$

$\exists v \in V(G) : d(v) \leq 5$

$P(k)$: we take $G - v$

From Kuratowski: deleting a vertex

cannot create a K.S.

$\rightarrow P(k)$ is planar

I.H. on $P(k)$ gives $P(k)$

a 5-coloring

Bring it back to $P(n)$

Case 1: $d(v) \leq 4$

$\rightarrow$ trivial to color

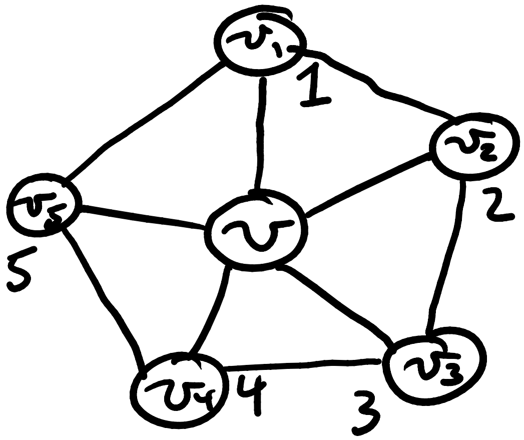
Case 2: $d(v) = 5$ and 4 or

fewer colors are in $N(v)$

$\rightarrow$ trivial to color

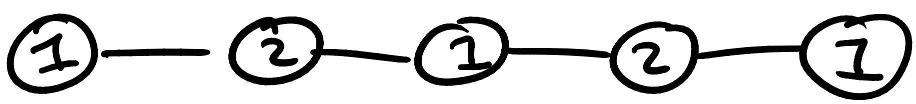
Case 3: $d(v) = 5$ and all

Case 3: $d(v) = 5$ and all of $N(v)$ will have a different color

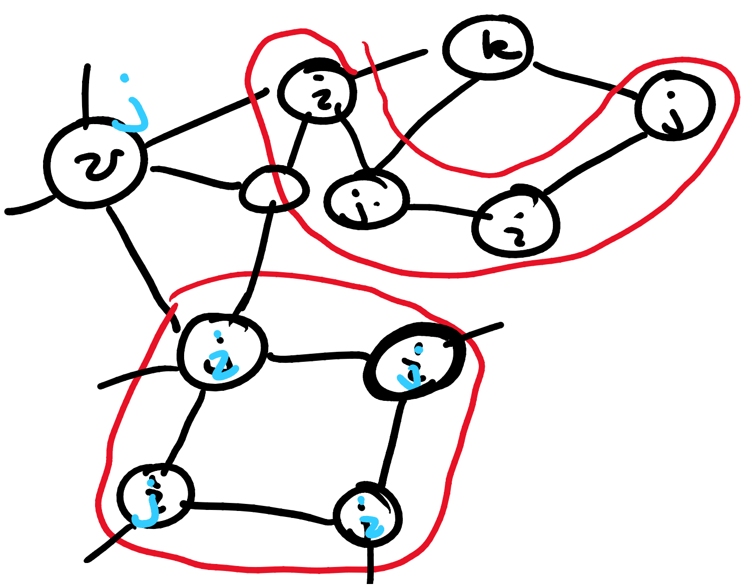


Show: this configuration can be "reduced" or colored with 5 colors

To do so: Kempe Chains aka color-alternating paths



Consider all possible Kempe chains around v for i,j color pairs



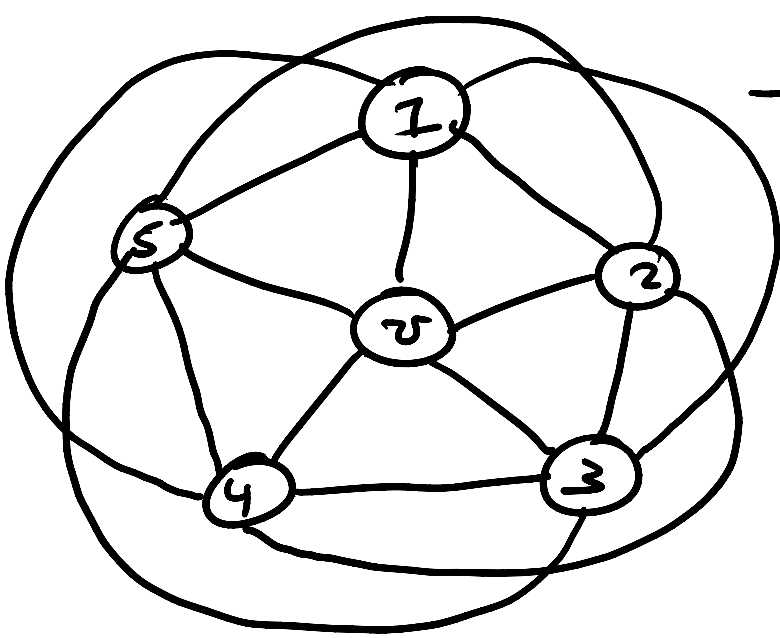
* if for a given i,j pair of colors, any i,j-alternating paths don't intersect

intersect

→ we can swap all i, j colors on one induced Kempe chain subgraph

AND color v with the color no longer in $N(v)$

Q: will there always be such an independent i, j pair of Kempe chains?



→ Note: if these paths all exist, we have a K_5 K.S.

contradiction

$\Rightarrow \exists$ at least one i, j pair that we can reduce $\square$

pair that we can reduce □

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what about 4-colors?

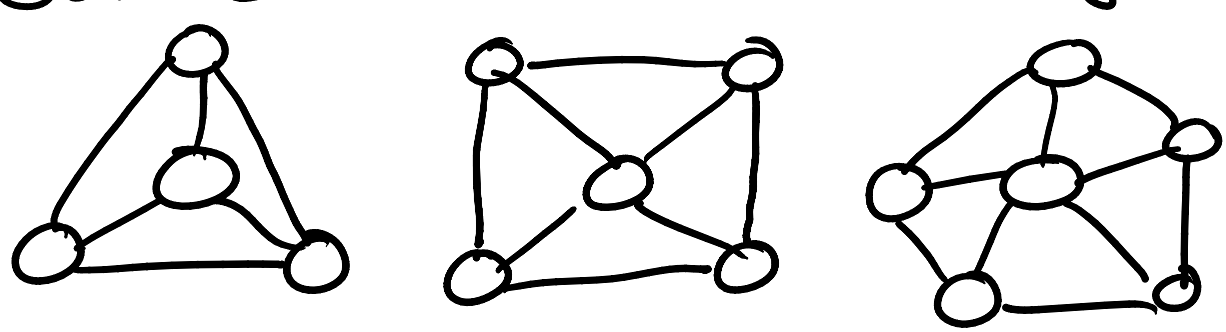
can we use the same approach?

→ We're looking to find some minimum unavoidable configuration that a counter-example must contain

→ If we can show all such possible configurations are reducible to color v with some color $1 \ldots 4$

$\Rightarrow$ all planar G are 4-color

Our 5-color theorem configurations



note: $\delta(G) \geq 3$ within a triangulation G

Let's try the same approach for 4 colors

Basis $P(\leq 4) \to$ trivial

$P(n)$: we have G, $v \in V(G)$ s.t. $d(v) \leq 5$

$P(k)$: $G - v$

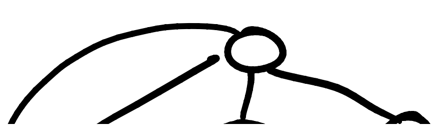
I.H. on $P(k) \to P(k)$ is 4-colorable

Bring it on back

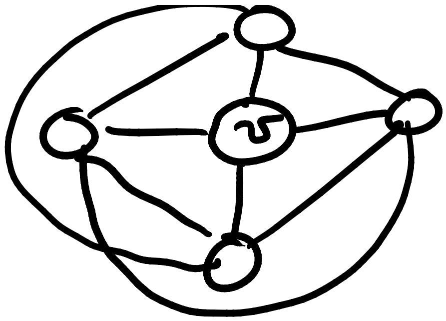
Case 1: $d(v) = 3 \to$ trivial

Case 2: $d(v) = 4$

We can use same argument as we did with 5-color theorem



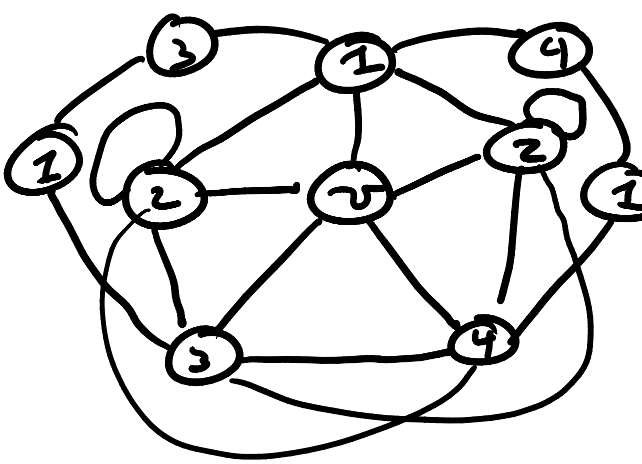
→ If some i, j pair



→ If some i, j pair we can't reduce exists, we have K_5 K.S.

Case 3: $d(v) = 5$

Note: exactly two vertices in $N(v)$ have the same color



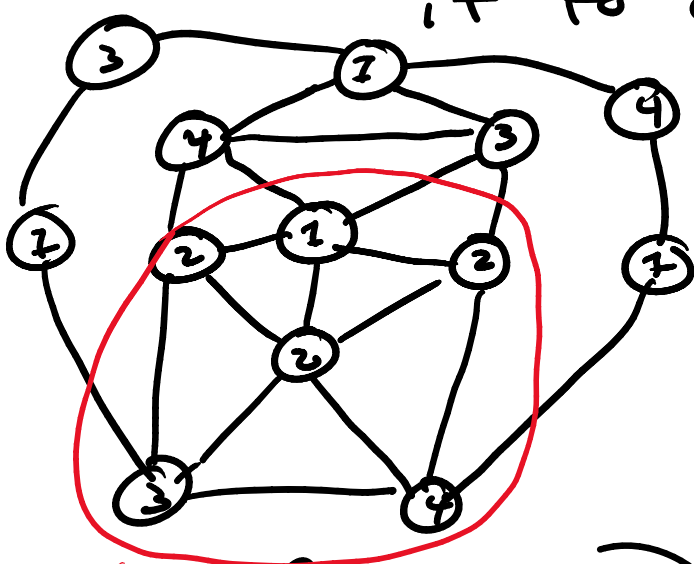
→ consider paths from $1 \to 3$ and $1 \to 4$

we can eliminate color 1 if these paths did not exist

Now consider vertices colored 2

As we can't have a 2-3 alternating path → we can swap colors of both of the induced subgraphs containing color 2 subgraphs containing color 2

$\rightarrow$ we can eliminate color 2 from $N(v)$ and use it to color v $\square$



NOPE

This is not the only configuration we need to consider

How many?

Originally: 1800

Now: 600

Our actual 4-color theorem proof: computationally determine all possible minimal configurations and their reductions