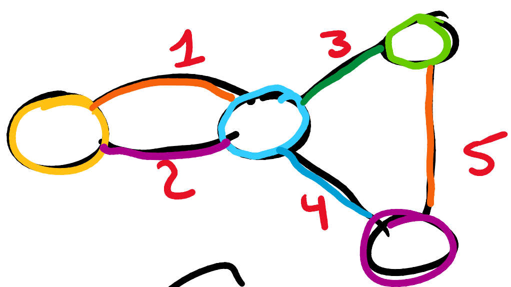
Lecture 22 - Line Graph and Edge Coloring

Thursday, April 13, 2023 3:57 PM

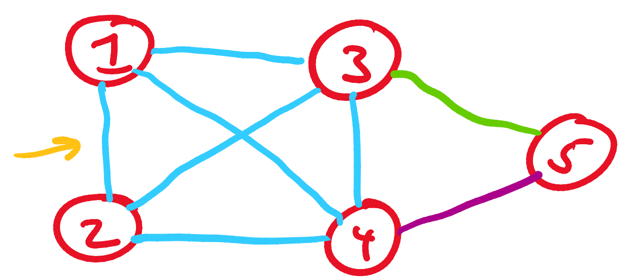
Line Graphs

The line graph of $G \to L(G)$

defined { edges of $G \to$ vertices of $L(G)$
edges of $L(G)$ exist where edges in G share an end point



G



$L(G)$

Note: each vertex in G corresponds to a clique $L(G)$

Note x2: the equivalence of $E(G) \leftrightarrow V(L(G))$ is relevant to several problems we've already discussed

1. Euler Tour on G

<=>

spanning cycle on L(G)

2. Matching on G

<=>

independent set on L(G)

3. cut edge on G

<=>

cut vertex on L(G)

4. edge-coloring on G

<=>

vertex coloring on L(G)

Edge - coloring

→ assigning labels to each edge in same G

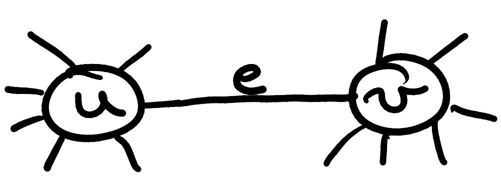
Proper: no two edges with the same label share an endpoint

Edge-chromatic number $\chi'(G)$

→ the minimum number of colors to properly edge-color graph G

Let's get Boundin'

$\chi'(G) \geq \Delta(G)$, as the largest degree vertex in G requires separate colors for all incident edges



$d(u) = \Delta(G) = d(v)$

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$\chi'(G) \leq 2\Delta(G) - 1$ via the worst case w/ greedy algorithm

$\chi'(G) = \Delta(G)$ if G is bipartite

Note: k-regular graphs have a perfect match

↳ all bipartite graphs are a subgraph of a k-regular graph (recall: planar $\subseteq$ triangulation)

↳ color our P.M. with one color, remove it, repeat until done

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we can actually tighten that upper bound by quite a bit

$\Rightarrow$ Show $\chi'(G) = \Delta(G)$

or $= \Delta(G) + 1$

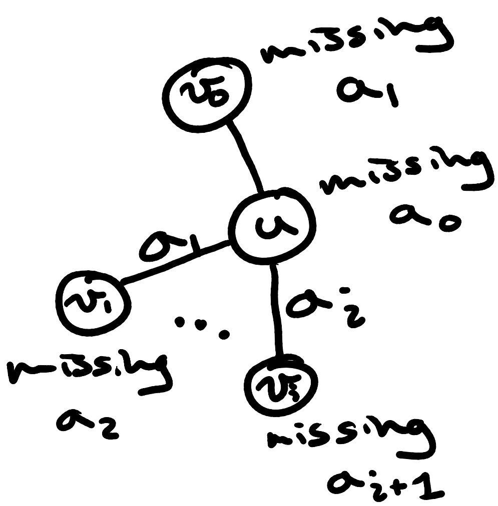
or $= \Delta(G)+1$
for simple G

PROOF BY ALGO.

Consider f as some $\Delta(G)+1$ edge-coloring of some subgraph $H \subseteq G \longrightarrow$ extend to all of G

consider $u \in V(G)$ and edge $(u, v_0) \in E(G)$ with no color

In $N(u)$, there are some colors missing $\rightarrow a_0$ is one such color



- consider u's neighbors
- label $N(u)$ s.t.

$a_{i+1} \rightarrow$ color missing at vertex v_i

If color a_0 is not in v_0's neighborhood $\rightarrow$ color (u, v_0) w/ a_0

If color a_1 is not in v_0's neighborhood and a_1 is not in u's neighborhood
$\rightarrow$ color (u, v_0) w/ a_1

If a_2 is missing at v_1, there must exist (u, v_2) with color a_2, otherwise we can just replace a_1 with a_2 and color (u, v_0) with a_1

Generally: if a_i is missing, we can use a_i on (u, v_{i-1}) and "shift" our colors down to eventually color (u, v_0) with a_1

$\rightarrow$ either a missing color repeats

or this procedure is possible

→ either a missing color repeats on this procedure is possible since we have at most $\Delta(G)+1$ edge colors

v_ℓ is the first vertex with a missing color on $a_1 \ldots a_\ell$

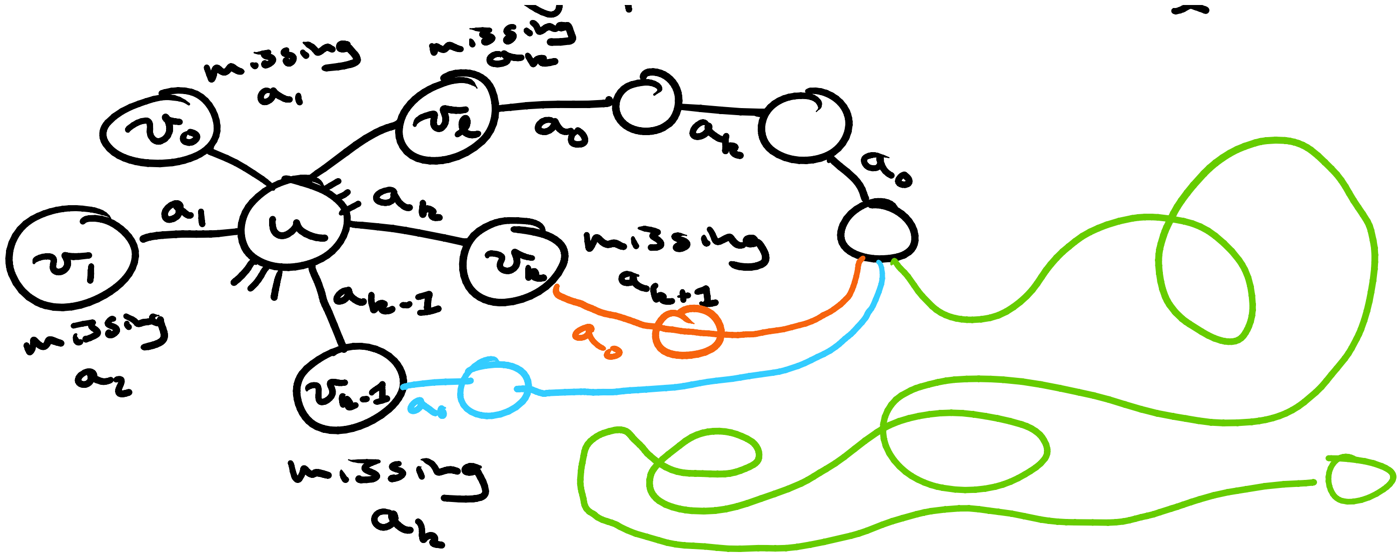
→ we'll just call it a_k

Note: also be missing at v_{k-1} and it is an edge (v_k, u)

Note x2: a_0 also appears on v_ℓ otherwise we could color (u, v_ℓ) with a_0 and shift the colors down

EXXXXTREMAL AGUMENT

Consider P as a maximal a_0, a_k alternating path from v_ℓ



Case 1: P reaches v_k

→ shift colors down from v_{k-1} and swap colors on P

Case 2: P reaches v_{k-1}

→ shift colors down from v_{k-1}, put a_0 on (u, v_{k-1}) and swap colors on P

Case 3: P reaches elsewhere

→ shift colors down from v_ℓ, put a_0 on (u, v_ℓ) and swap colors on P

=> no matter what, all graphs have a

all graphs have a edge-chromatic number of

$\Delta(G)$ or $\Delta(G)+1$ $\boxed{\checkmark}$