Lecture 23 - More Line Graphs

Monday, April 17, 2023 12:06 PM

As we've discussed last class:

if $\exists H$ s.t. $L(H) = G$

- → we can get a maximum indep. set in polynomial time (normally exponential)
- → we can get an optmal vertex coloring in quadratic time

Note: don't worry about complexity stuff for the final (non-CS students kept getting zeros across the board)

Our Q: given some G, does there exist some H s.t. G is the line graph of H?

If for all G, if there was same $H \to P = NP$

Q2: For what G does such an H exist?

$$L(H) = G$$

From our example
→ a line graph can decompose into maximal cliques with each vertex in at most 2

So the above condition must hold for any G s.t. $L(H) = G$

Let's prove this as an equivalence

→ For simple G, $\exists H$ s.t. $L(H) = G$ iff G decomposes into iff G decomposes into maximal cliques with each $v \in V(G)$ in at most 2

($\Rightarrow$)

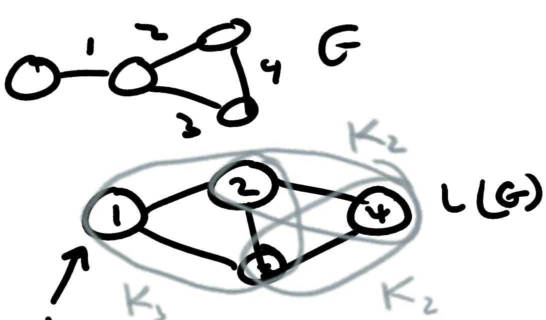
Note: every vertex in H becomes a clique in G

And: every edge in H is attached to at most 2 vertices in H → cliques in G

edges in H → vertices in G ✓

($\Leftarrow$)

define: $S_1, S_2 \dots S_k$ as vertex sets of maximal cliques in a decomposition of G



only in 1 K_n

To construct H:

$v_1 v_2 \dots v_e$ are vertices in only 1 of S_i

$V(H) = \{$one vertex for each in $\{S_1 S_2 \dots S_n\}, \{v_1 v_2 \dots v_e\}\}$

$E(H) = \{$ for all (v_i, S_j) and (S_n, S_m) where these vertizes intersect$\}$

$\rightarrow$ each $v \in V(H)$ is in at most two sets S_i with no two vertizes in the same 2 sets

$\Rightarrow$ together, this implies the existence of our H s.t. $L(H) = G$ $\square$

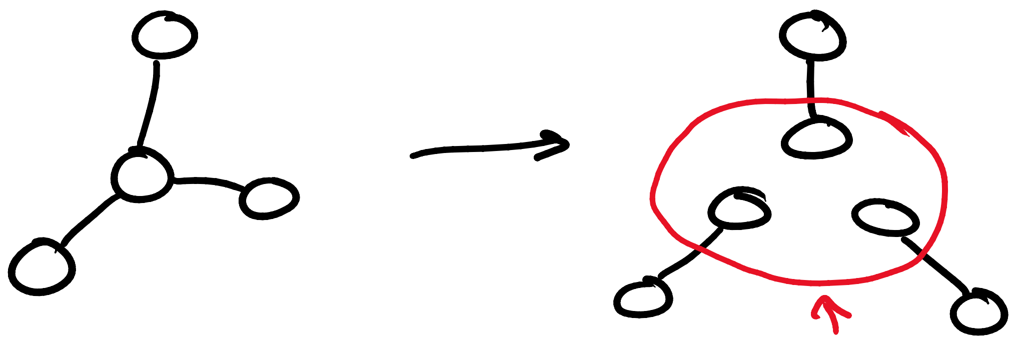
Big Q:

Is there an easier characterization

Is there an easier characterization of G for when $\exists H$ s.t. $G = L(H)$?

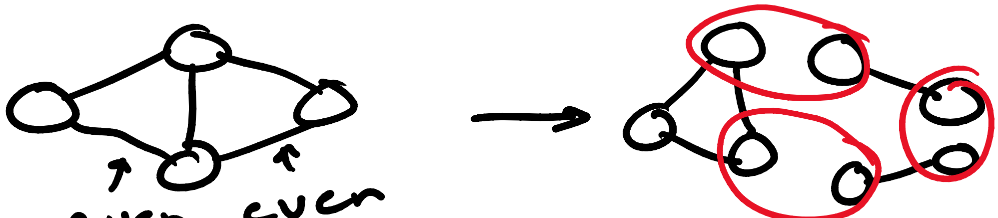
A: yes

Consider an induced claw (aka $K_{1,3}$)



$\Rightarrow$ no such G can have an <u>induced</u> claw as a subgraph

Consider a double triangle



what about:

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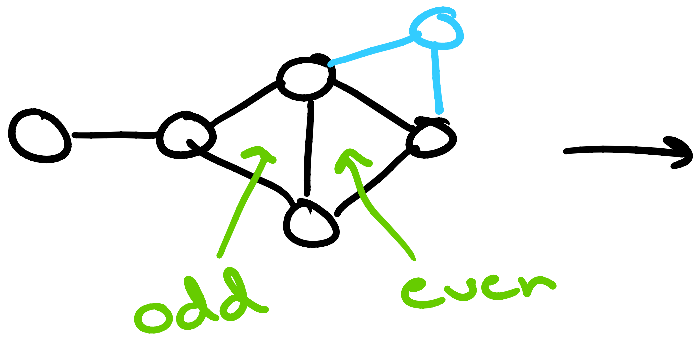


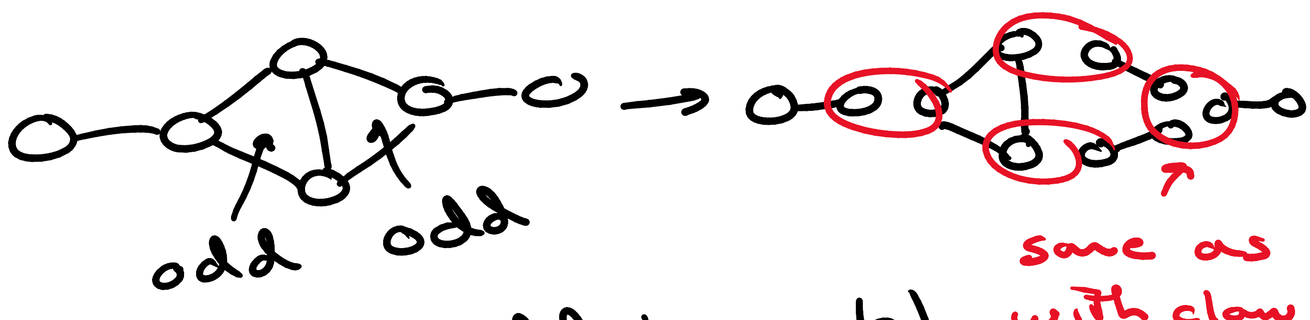
odd triangle T: $\exists v \in V(G), T \subseteq G$

s.t. $|N(v) \cap V(T)| = \text{odd}$

even triangle T: $\forall v \in V(G), T \subseteq G$

s.t. $|N(v) \cap V(T)| = \text{even}$

Now consider:



(double odd triangle)

$\Rightarrow$ no such G can have a double odd triangle as double odd triangle as a subgraph

As before: we've demonstrated necessity, but are these conditions also sufficient

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$\exists H$ s.t. $G = L(H)$ iff

G has no claws or double odd triangles (DOTS)

($\Rightarrow$) Contrapositive

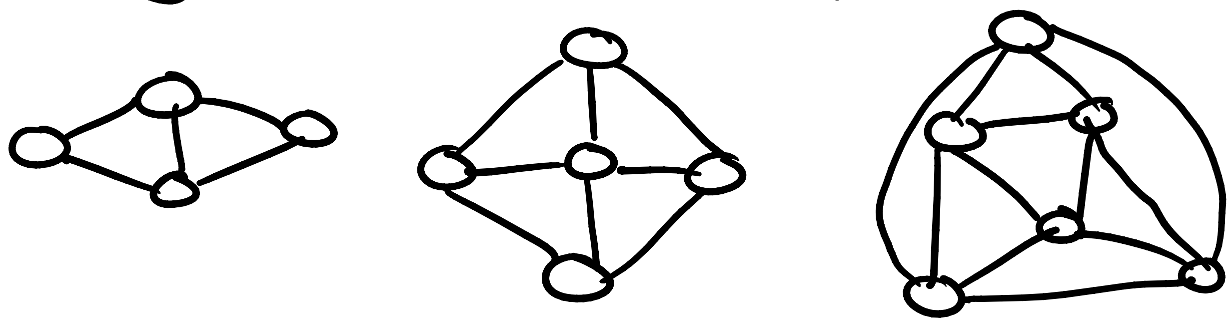
G has claws or DOTS $\Rightarrow$ no H

$\rightarrow$ we just showed this ✓

($\Leftarrow$)

First: consider double even triangles

→ only 3 exist for simple graphs



⇒ we only need to consider graphs with double triangles that have one odd and one even triangle

consider a maximal clique decomposition, with one special caveat:

$S_1, S_2 \ldots S_k$ are maximal cliques <u>except</u> for even triangles that aren't shared with an odd triangle

Q: $\forall v \in V(G)$, is v in at

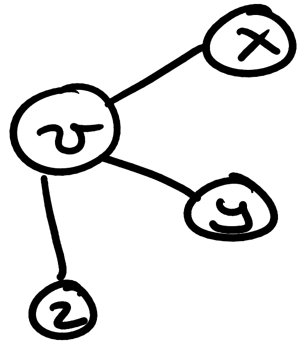
Q: $\forall v \in V(G)$, is v in at most 2 subgraphs in our decomposition?

consider $v \in S_i S_j S_k$

$\{x, y, z\} \in N(v)$

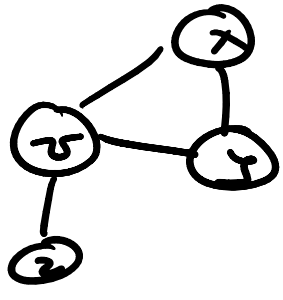
$x \in S_i$, $y \in S_j$, $z \in S_k$

Case 1: no edges (x,y) (y,z) (x,z)



→ a claw, so this configuration can't exist

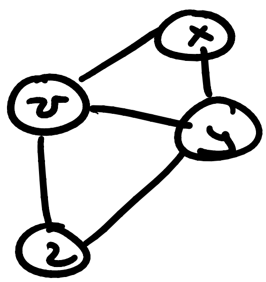
Case 2: edge (x,y) exists



→ an odd triangle, but our decomposition specified odd triangles in a single S_e

Note: for all $K_n : n > 3$, they are comprised of odd triangles

Case 3: edges (x,y) and (y,z) exist



→ we have two even triangles, regardless of choice of v

Case 4: edges $(x,y)(y,z)(x,z)$ exist

→ we have K_4, which would be in only one of S_e

⇒ taken together, along with our assumed decomposition, v can be in at most two subgraphs in that decomposition

$\Rightarrow \exists H$ s.t. $G = L(H)$ $\square$

Our characterization of G

G has no DOTS

G has no claws

Forbidden subgraphs

Forbidden subgraphs

Take away:

$\exists H$ s.t. $G = L(H)$ iff G has no forbidden subgraphs