Lecture 24 - Hamiltonian Graphs

Thursday, April 20, 2023 3:55 PM

Q: What is a Hamiltonian graph?

A: A graph with a spanning cycle

Hamiltonian cycle

$\iff$

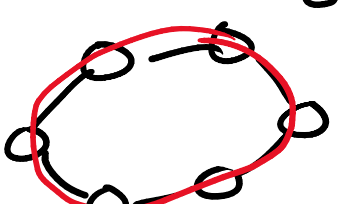
spanning cycle

↳ a subgraph $C \subseteq G$ ← cycle

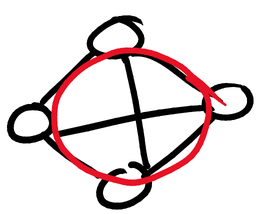
$|V(C)| = |V(G)|$

$C \to C_{|V(G)|}$ cycle graph

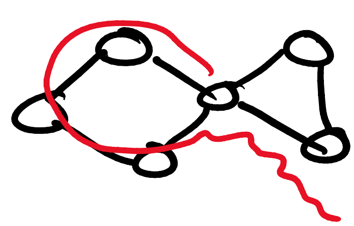
What graphs are Hamiltonian?



cycle graphs



cliques



Eulerian graphs?

cycle graphs | cliques $n \geq 3$ | Eulerian graphs: → not necessarily

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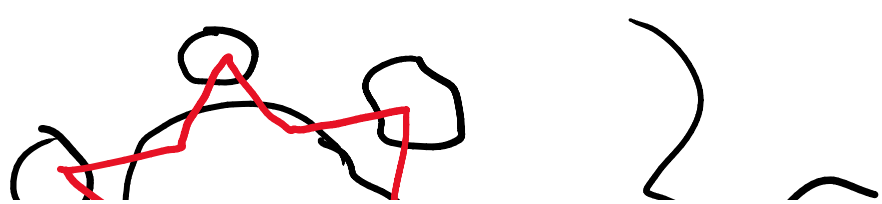
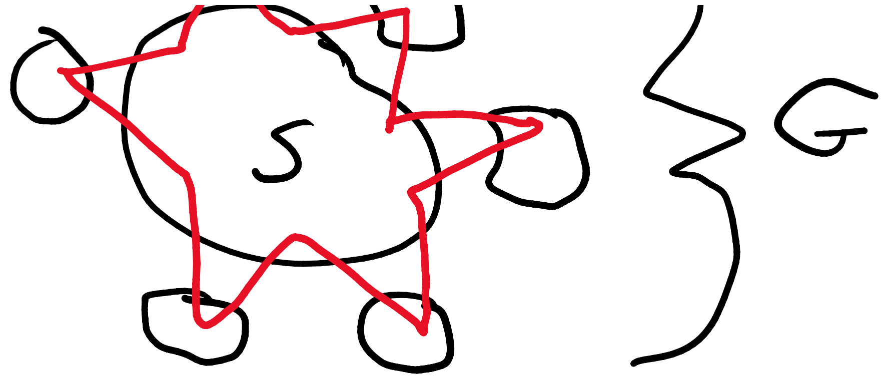
Necessary conditions for G to be Hamiltonian

- G must be connected
- G must be 2-connected
  → a spanning cycle can't pass through a cut vertex
- If G is bipartite with X, Y sets then $|X| = |Y|$
  → a cycle traverses between X and Y an equal number of times
- If $c(G)$ is # components of G, then $c(G-S) \leq |S| \ \forall S \subseteq V(G)$





→ For every component, we need a unique vertex in S from which to traverse

These conditions are necessary.

Q: but what about sufficient conditions?

Sufficient conditions for Hamiltonian graphs

if $|V(G)| \geq 3$ and $\delta(G) \geq \frac{|V(G)|}{2}$

Consider maximum non-Hamiltonian G'

→ $G' + e$ = Hamiltonian

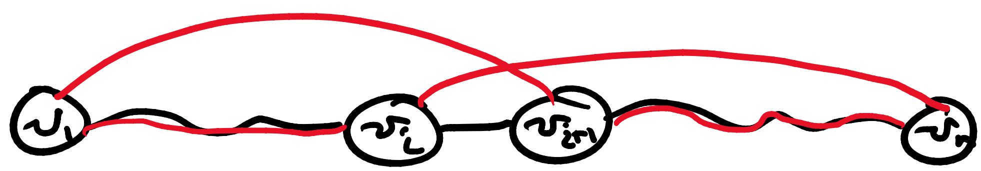
→ G' has a Hamiltonian path

→ G' has a Hamiltonian path
↳ spanning path

Consider this path in same order $P = \{v_1, v_2 \dots v_n\}$

If along this path $\exists\, v_i v_{i+1}$ s.t. $v_i \in N(v_n)$, $v_{i+1} \in N(v_1)$

→ we can construct a Ham. cycle



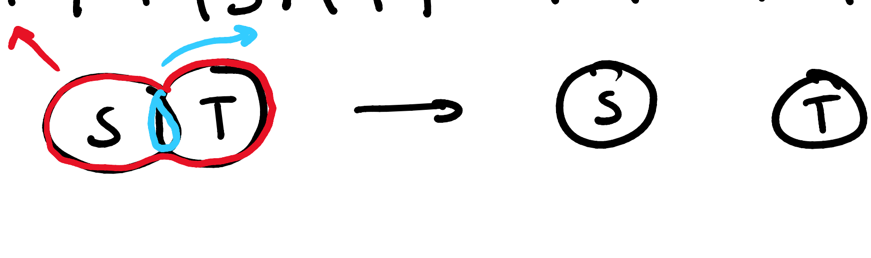
define $S = \{ i : (v_1, v_{i+1}) \}$

$T = \{ i : (v_n, v_i) \}$

show $|S \cap T| \geq 1$

→ we have a cycle

$|S \cup T| + |S \cap T| = |S| + |T|$



$$|S| + |T| = d(v_1) + d(v_n) \geq |V(G)|$$

$$|S \cup T| + |S \cap T| \geq |V(G)|$$

$|S \cup T| < |V(G)|$ as we assume no $(v_1 v_n)$ edge

$\hookrightarrow |S \cap T| \geq 1$

$\Rightarrow$ we have a spanning cycle

Applying this to all pairs of vertices constructs our spanning cycle in the general case $\square$

If $\forall u, v \in V(G)$ $(u,v) \notin E(G)$
$d(u) + d(v) \geq |V(G)|$

G is Hamiltonian iff
$G + (u,v)$ is Hamiltonian

$(\Rightarrow)$ trivial, as adding an edge won't delete a spanning cycle

[illegible]
won't delete a spanning cycle

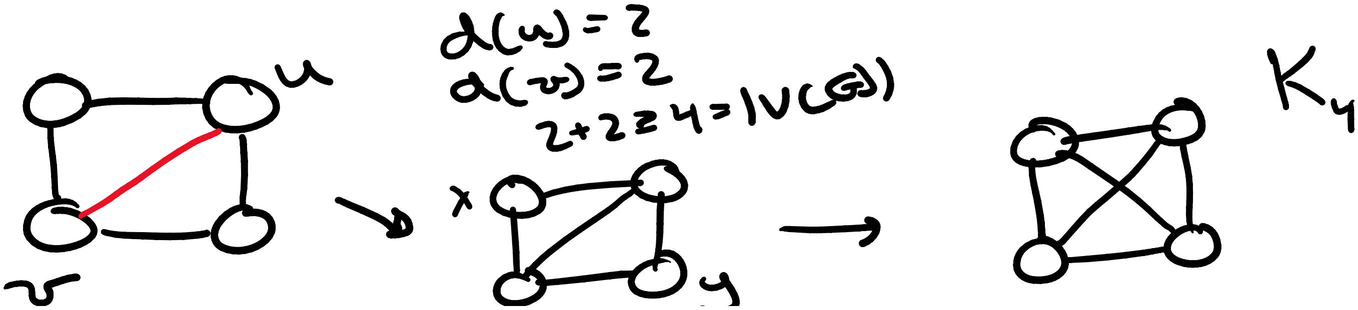
($\Leftarrow$) this follows from our prior proof since $|N(u) \cap N(v)| \geq 1$

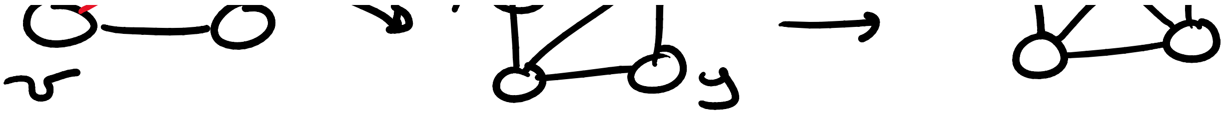
We can use the above to determine the <u>closure</u> of G

closure of G:
add (u,v) to $E(G)$
$\forall u,v \in V(G)$
s.t. $d(u) + d(v) \geq |V(G)|$

From above:
If G's closure is Hamiltonian
$\iff$ G is Hamiltonian

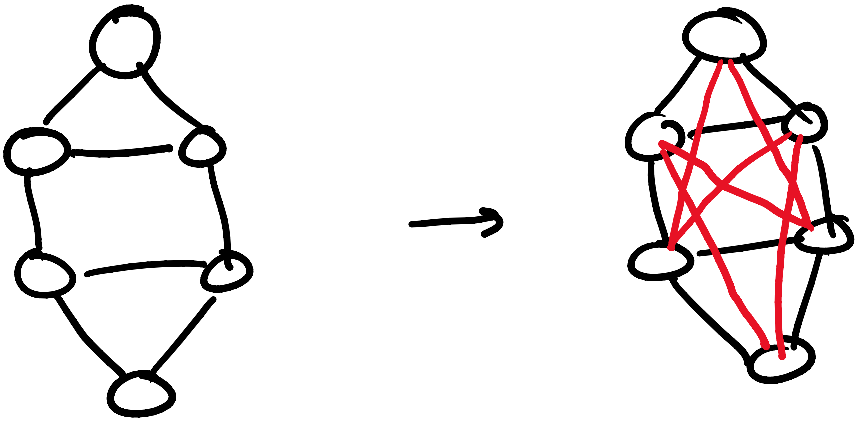
Let's look at some closures





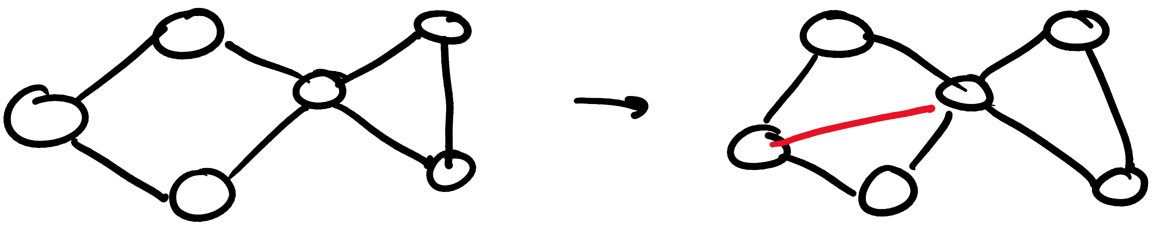
G $\quad$ $G+(u,v)$ $\quad$ $G+(u,v)+(x,y)$

(closure of G)



G

$K_6 \rightarrow$ Hamiltonian



G

closure of G

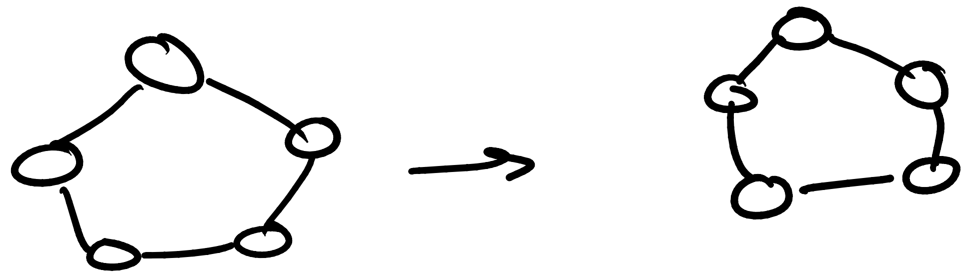
→ <u>not</u> Hamiltonian

E2

sufficient condition:

If G's closure is a clique, the G is Hamiltonian

Note: above is not necessary



G closure of G

Q: is the closure of G well-defined?

Consider

$e_1, e_2 \ldots e_i$ and $f_1, f_2 \ldots f_j$ are edges added to create the closures of $G \to G_e, G_f$

→ since e_1 can be added for G_e, it must also be added for G_f as some f_k

→ If any e_ℓ depends on e_1, there is equivalently some f_m that depends on f_k and will be added to G

s_m that depends on s_k will
<u>will</u> be added to G_f

→ consider this for all $e_1 \dots e_i$
and note they'll all be
added to G_f and vice-versa
(vis-versa)
(sp?)

=> all the same edges will
be added to both G_e
and G_f → $G_e \cong G_f$ ☑

Instead of explicity
constructing a closure and
checkin if its a clique

→ we can define a
numerical relation on
the degree sequence
of G

=> Chvátal's Condition

consider G with degrees

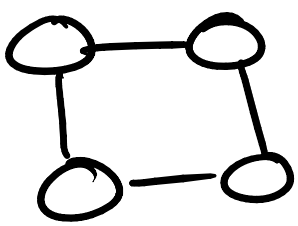
$d_1 \le d_2 \le \dots \le d_n$

if $i < \frac{n}{2}$ implies $d_i > i$

or $d_{n-i} \ge n-i$

→ Closure of G is a clique

=> G is Hamiltonian



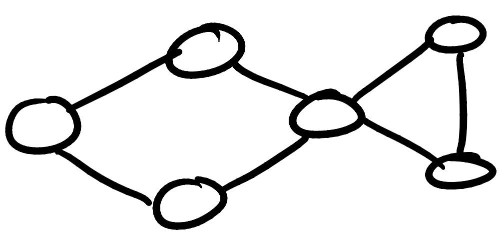
$S = 2\ 2\ 2\ 2$

$i = 1\ 2\ 3\ 4$

$n = 4$

$i = 1$

$d_1 = 2 > 1$ ✓



$S = 2\ 2\ 2\ 2\ 2\ 4$

$i = 1\ 2\ 3\ 4\ 5\ 6$

$n = 6$

$i = 1$

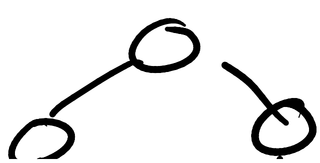
$d_1 = 2 > 1$ ✓

$i = 2$

$d_2 > 2 > 2$ ✗

$d_{n-i} = d_4 = 2$

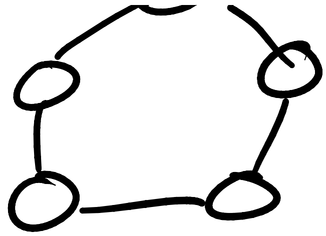
condition doesn't hold $\quad d_4 = 2 \ge 4$ ✗



$S = 2\ 2\ 2\ 2\ 2$

$i = 1$

$d_1 = 2 > 1$ ✓



$S = 2\,2\,2\,2\,2$

$i = 1\,2\,3\,4\,5$

$n = 5$

$i = 1$

$d_1 = 2 > 1$ ✓

$i = 2$

$d_2 = 2 > 2$ ✗

$n - i = 3$

$d_3 = 2 \geq 3$ ✗

Note: Chvátal's condition doesn't hold but G is Hamiltonian

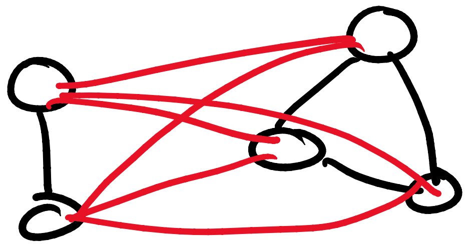
→ Chvátal's condition is <u>sufficient</u> but not <u>necessary</u>

Hamiltonian path → spanning path

<u>Graph join</u> between G and H, notationally as $G \vee H$, is an edge (u, v) $\forall u \in V(G)$ $\forall v \in V(H)$

Join K_2 and K_3

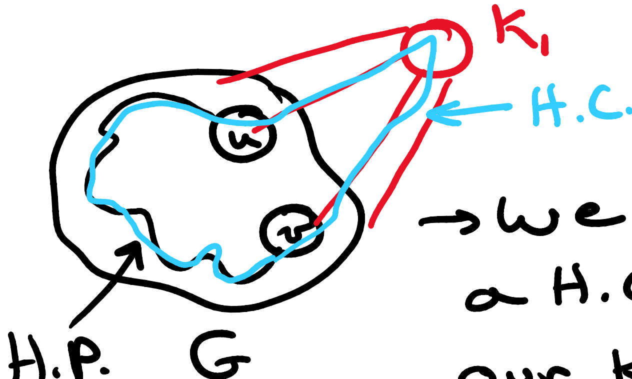
join K_2 and K_3



K_2 K_3 $K_2 \vee K_3 \cong K_5$

G has a Hamiltonian path iff $G \vee K_1$ has a Hamiltonian cycle

($\Rightarrow$)



$\rightarrow$ we can construct a H.C. on $G \vee K_1$ through our K_1 vertex and the endpoints of the assumed H.P.

($\Leftarrow$) Necessarily, exactly 2 of K_1's edges are going to be a part of the assumed H.C.

$\rightarrow$ we can delete them and

→ we can delete them and a H.P. remains on G □

When considering the above, we can modify Chvátal's condition for Hamiltonian paths

If G has degrees

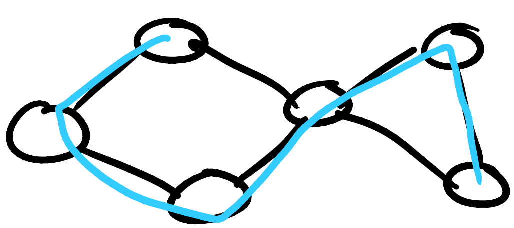
$d_1 \leq d_2 \leq \dots \leq d_n$

then if

$i < \frac{n+1}{2}$ implies $d_i \geq i$

or $d_{n+1-i} \geq n-i$

$\Rightarrow$ G has a Hamiltonian Path □



$S = 222224$

$i = 123456$

$n = 6$

$i = 1$

$d_1 = 2 \geq 1$ ✓

$i = 2$

$d_2 = 2 \geq 2$ ✓

$i = 3$

Note: also <u>sufficient</u>

Note: also sufficient but not necessary

$i = 3$

$d_3 = 2 \geq 3$ ✗

$d_4 = 2 \geq 3$ ✗