Final Practice - Solutions

Sunday, April 21, 2024 12:00 PM

① G has no even cycles and is 1-connected

Q: can G have some B_i which is not $C_{n=odd}$ or K_2?

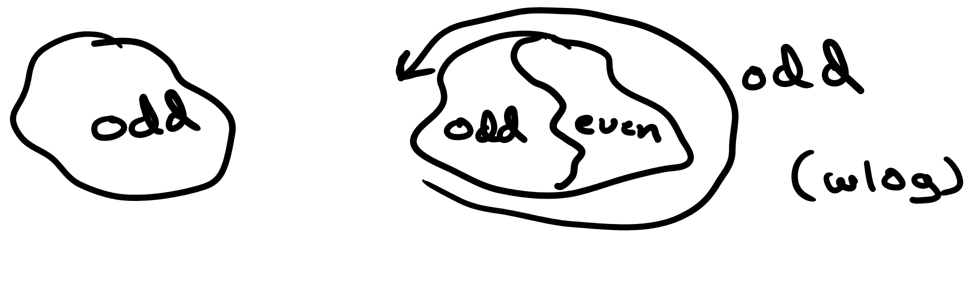
→ we build a closed-ear decomp

C_0 = any cycle in G

(no cycles → G is tree all $B_i \cong K_2$)

Is there an open ear on C_0?

→ No, since adding an ear to an odd cycle "splits" the cycle into one odd and one ~~even~~ cycle



→ So any ear must be from

→ So any ear must be from a cut vertex

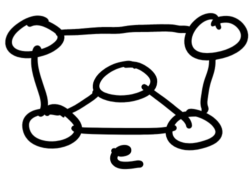
↳ Note: ear not possible if K_2

↳ Otherwise, ear must be odd cycle

Repeat above logic

⇒ we only have K_2 and odd cycles in G □

(2)



$|V(G)| = 5 \geq 5$

$\chi(G) = 3 = \omega(G)$

$\chi(G-e) = 3 \neq \omega(G-e) = 2$

(3) Color some $C_{odd} \subseteq G$ with 3 colors

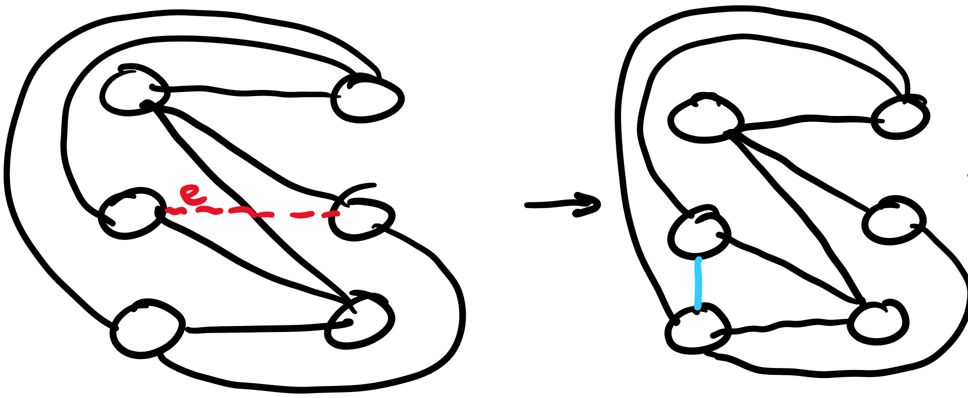
→ $G - V(C_{odd})$ is bipartite, as we delete a vertex from every odd cycle

→ we can color $G - V(C_{odd})$ with at most 2 colors

$\Rightarrow G$ is $3+2=5$-colorable $\square$

④ Disproof via eggzample

→ consider $K_{3,3} - e + f$



⇒ not maximal planar

Note: w.l.o.g. we can select any e since they are all automorphically equivalent $\square$

⑤ a) Has closed ear → $K'(G) \geq 2$

Has no open ear → $K(G) = 1$ (but is connected)

$\delta(G) = 4 \rightarrow K'(G) \leq 4$

$\Rightarrow \boxed{K(G) = 1,\ 2 \leq K'(G) \leq 4}$

b) Assuming "G is not biconnected"

Not biconnected → $K(G) \leq 1$

Has a block of $K_4 \to K'(G) \leq 3$

(delete edges of some $v \in K_4$)

Can't say if G connected

$\Rightarrow \boxed{K(G) \leq 1,\ K'(G) \leq 3}$

6 a) Solve for smallest k s.t. $\chi(G,k) > 0$

$\chi(G,1) = 0 - 0 + 0 = 0$

$\chi(G,2) = 2 - 2 + 0 = 0$

$\chi(G,3) = 48 - 24 + 3(2)(1) = 30 > 0$

$\Rightarrow \boxed{\chi(G) = 3}$

b) $\omega(G) \geq 4 \to \chi(G) \geq 4$

No other given info is relevant

$\Rightarrow \boxed{\chi(G) \geq 4}$

7 Outerplanar G is 3-colorable

Note: We want to find induced

Note: We want to find induced subgraphs G_1, G_2 such that $V(G_1) \cup V(G_2) = V(G)$ and $\Delta(G_1) \leq 2$, $\Delta(G_2) \leq 2$ and G_1 and G_2 have no cycles

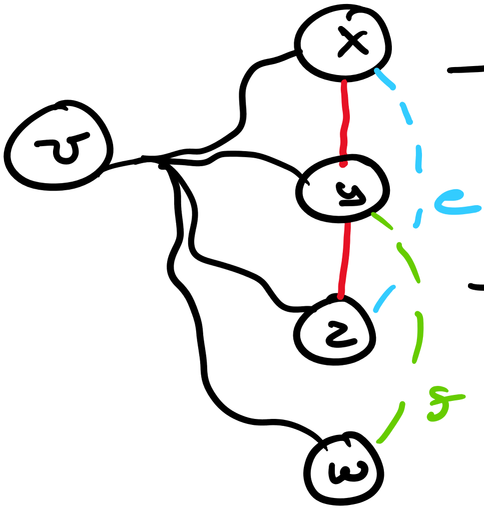
Consider some $v \in V(G)$ and a BFS from v tracking distance

→ some $x, y \in V(G)$ cannot have edge (x, y) unless they are within one level of the BFS

→ We'll create G_1 and G_2 by inducing on every other level of the BFS tree (even/odd distance from v)

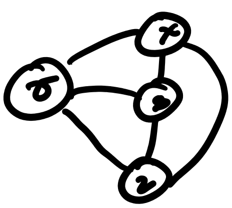
Now consider some $x y z \in V(G_1)$ that were all the same distance from root v

(Note: each level creates a separate component)

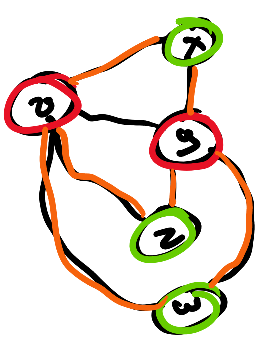


→ assume (x,y) and (y,z) exist w.l.o.g.
(only 2 verts or disconnected trivial)
→ x, y, z and their shortest paths to v form a claw subdivision

→ edge $e = (x,z)$ cannot exist to create a cycle, since that creates a K_4 subdivision, which an outerplanar graph cannot have



→ edge $f = (y,w)$ to create degree-3 vertex y is not possible, since that creates a $K_{2,3}$ subdivision with the claw



=> we must have no cycles or

$\Rightarrow$ we must have no cycles or vertex degrees greater than 2 in G_1 or G_2 and hence are a disjoint union of paths $\square$

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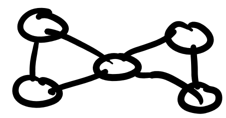
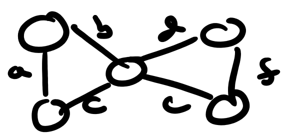
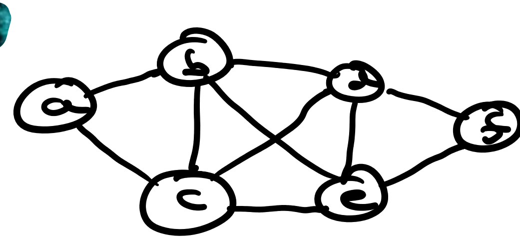
a) Counter-example: 

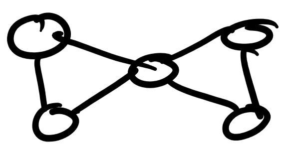
b) same counter-example works

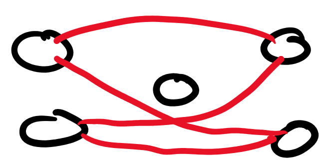




G $\quad$ $L(G)$

c) one more time





G $\quad$ $\overline{G}$

d) We proved in class

G is Hamiltonian $\Leftrightarrow$ closure of G is Hamiltonian

$\Rightarrow G$ is Hamiltonian

e)

∴ G is Hamiltonian

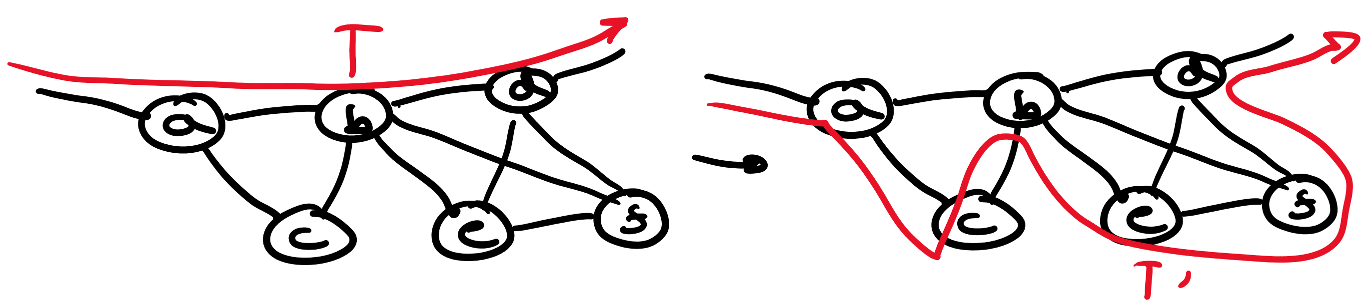
e) Consider extrapolating the Ham. cycle from $H \to G$

→ this cycle becomes a closed trail T on G, following edges →verts

→ on H, the cycle also covers all edges

→ So we are incident to any clique containing a "missed" vertex in T on G

We can simply modify T to visit any such missed vertices



→ When we encounter a clique with a missing vertex, we construct T' by detouring to visit those vertices

$\Rightarrow$ we get a cycle on $L(G)$ $\square$