Homework 1 - Solutions

Sunday, February 4, 2024 6:10 PM

① a) we can "rotate" our vertex labels across n choices and "mirror" our labels over 2 choices

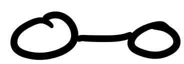
$\Rightarrow$ $2n$ automorphisms

b) as we saw in class, all vertices can assume all possible choices

$\Rightarrow$ $n!$ automorphisms

c) we can only mirror our labels over 2 choices

$\Rightarrow$ 2 automorphisms

② $S = \{1\ 1\} \Rightarrow$ 

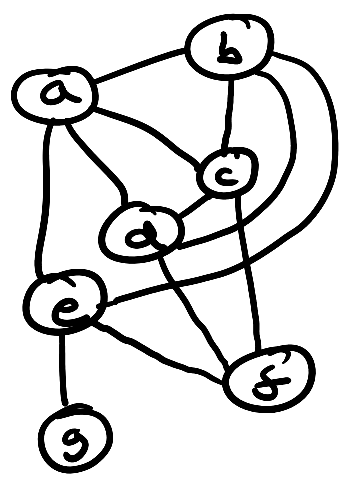
$L = \{\overset{a}{\cancel{4}}\ \underset{-1}{\overset{b}{4}}\ \underset{-1}{\overset{c}{4}}\ \underset{-1}{\overset{d}{4}}\ \underset{-1}{\overset{e}{4}}\ \overset{f}{3}\ \overset{g}{1}\}$

$L' = \{\overset{b}{\cancel{3}}\ \underset{-1}{\overset{c}{3}}\ \underset{-1}{\overset{d}{3}}\ \underset{-1}{\overset{e}{3}}\ \overset{f}{3}\ \overset{g}{1}\}$

$L'' = \{\overset{f}{\cancel{3}}\ \underset{-1}{\overset{c}{2}}\ \underset{-1}{\overset{d}{2}}\ \underset{-1}{\overset{e}{2}}\ \overset{g}{1}\}$

$L''' = \{\overset{c}{\cancel{1}}\ \underset{-1}{\overset{d}{1}}\ \overset{e}{1}\ \overset{g}{1}\}$

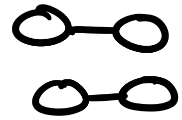
$L'''' = \{\overset{e}{\cancel{1}}\ \underset{-1}{\overset{g}{1}}\ 0\}$



$O = \{7\ 5\ 2\ 1\ 3\ 1\}$

degree too large => no realization

$T = \{1\ 1\ 1\ 1\}$ =>



$A = \{1\ 2\ 3\ 4\ 5\}$

degree too large => no realization



③ If $G$ is a multigraph, it must have $C_2$

First, assume $G$ is connected (and simple)

→ we know trees are maximally acyclic and have $|V(G)| = |E(G)| + 1$

=> Hence, our $G$ must have at least two more edges than a tree, so it must have a cycle

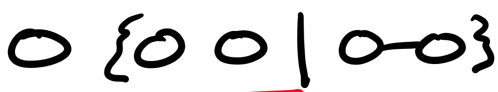
If $G$ is disconnected, it will have some $k$ components

$S_1\ S_2 \dots S_k : S_i \subseteq G$

$|V(S_1)| + \dots + |V(S_k)| = |V(G)|$

$|E(S_1)| + \dots + |E(S_k)| = |E(G)|$

Hence, for at least one $S_i$, we have $|E(S_i)| > |V(S_i)|$ and the above argument holds for component $S_i$, giving us a cycle in $G$ $\square$

④ a) 

n=1 $\boxed{n=2}$ (isomorphism classes)

b) 

n=1 $\boxed{n=2}$

c) 

n=1 $\boxed{n=2}$

d)  ← infinite loops

$\boxed{n=1}$

⑤ Strong induction on $|E(G)|$

$P(0)$ : empty graph, trivially bipartite ✓

Assume we have some $P(n)$ with no odd cycles

$P(k) = P(n) - e$, edge deletion can not create a cycle

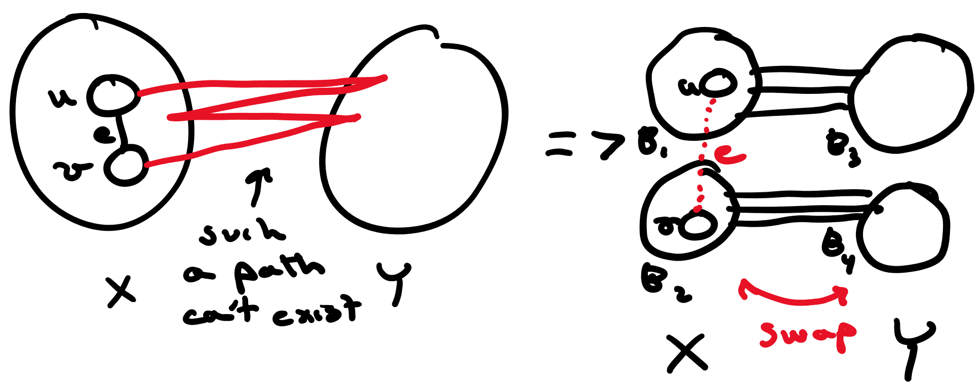
I.H. on $P(k)$ gives us a $X, Y$ bipartition of $V(P(k)) = V(P(n))$

We add back edge $e = (u, v)$

Case 1: $u \in X$, $v \in Y$, the bipartition on $P(n)$ is the same

Case 2: $u \in X$, $v \in X$ wlog

$\rightarrow$ this implies that $e$ is a cut edge in $P(n)$, otherwise $e$ is on an odd cycle




We'll define:

$B_1$ = subgraph of $P(k)$ containing $u$ in bipartition $X$

$B_2$ = subgraph of $P(k)$ containing $v$

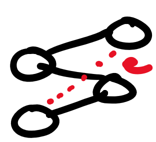
in bipartition X

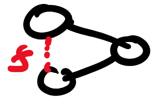
$B_2$ = subgraph of $P(n)$ containing $v$ in bipartition X

$B_3$ = part of Y in same component as $B_1$

$B_4$ = part of Y in same component as $B_2$

We can get a valid bipartitioning on $P(n)$ by mapping all of $V(B_2)$ to Y and all of $V(B_4)$ to X $\square$

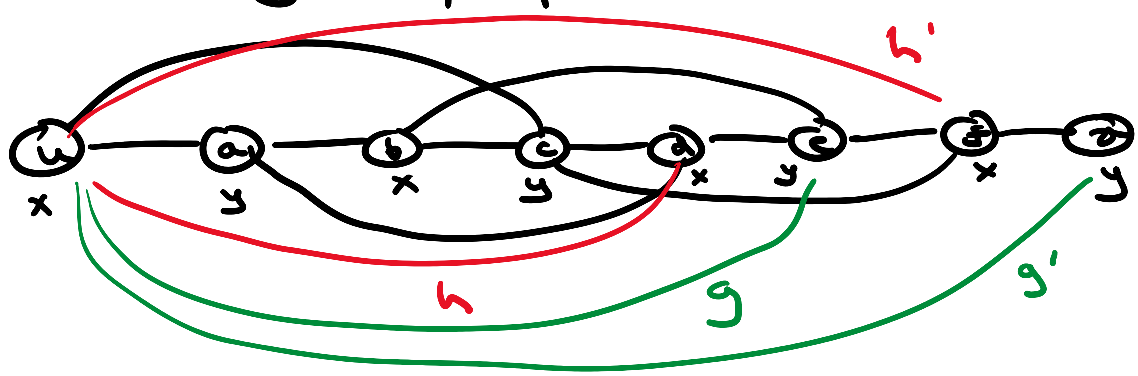
(6) Note: G cannot have a subgraph $P_4$ , so edge e must always exist

Note 2: G cannot have a triangle $C_3$ , so edge f must never exist

Note 3: G is connected, so

Note 3: G is connected, so all u,v have a u,v-path

=> along any u,v-path, every subpath of $P_4$ has an edge closing it, per Note 1




We first observe that edge g must exist, as otherwise we would have open $P_3 = \{u, a, d, e\}$ and violate Note 1

→ Further, we would need all possible odd paths closed, otherwise we violate Note 1

See how the existence of g will create an open $P_3$ on (uefv) without edge g' without edge $g'$

Likewise edges $h$ and $h'$ cannot exist, as with the above $g, g'$ will result in a triangle and violate Note 2

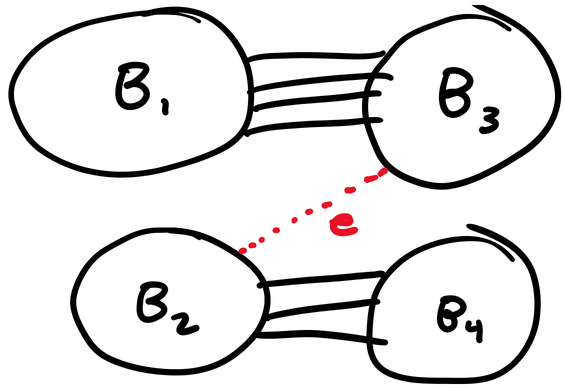
→ So we cannot have any odd cycles AND on any $u,v$ path *all* vertices an odd distance from $u$ have an edge to $u$

⇒ Hence, we have a connected bipartite graph where any $u \in X$ is connected to all $v \in Y$, also (likewise for $u \in Y$ to $v \in X$) known as a biclique □

7 In bipartite $G_{XY}$: $\sum_{u \in X} d(u) = \sum_{v \in Y} d(v)$

If $G$ is k-regular: $|X| = |Y|$

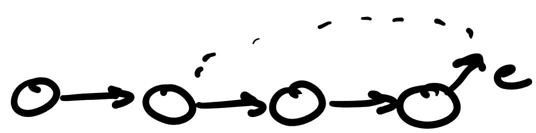
Consider the below configuration if a cut edge existed on $G$




Per the above, the sum of the degrees of $B_1$ equals $B_3$ necessarily, but edge e "takes away" exactly 1 from the degree sum of $B_3$, which is a contradiction

=> Hence, no such e exists □

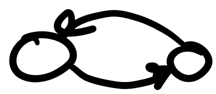
8 a) Consider a maximum path on G

P 


edge e must connect to some prior vertex in P, so at least one cycle must exist

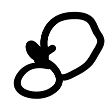
We also note that at most one cycle can exist, as each vertex has an out edge, so the graph is a DAG up until one final v is a DAG up until one final $v$ on a maximal path that connects to a prior vertex in a topological order $\square$

b) The smallest component of $O$ is of size 2 and is a $C_2$ cycle



Hence, we maximize the number of cycles by maximizing # comps, giving us $\lfloor \frac{n}{2} \rfloor$ cycles

c) Similar argument:

$C_1$ is smallest comp 

and $n$ is max possible cycles $\square$