Graph Theory Homework 1
Due: 2 Feb 2024 at midnight EST as a PDF on Submitty
v1.1: Last Updated January 21, 2024

1. Determine the number of automorphisms for each of the following graphs, in terms of n:
   (a) Cycle graph C_n
   (b) Clique graph K_n
   (c) Path graph P_n
2. Which of these sequences are graphic? For the ones that are, construct a realization. $S = \{1,1\}, L = \{4,3,4,4,4,1,4\}, O = \{7,5,2,1,3,1\}, T = \{1,1,1,1\}, A = \{1,2,3,4,5\}$
3. Consider graph G where $|V(G)| < |E(G)|$. Prove that G must have a cycle. Note: I don't give any additional assumptions about G, so you will need to explicitly consider if G is connected, simple, etc.
4. Identify the smallest n for the classes of graphs defined below, such that there exists two non-isomorphic graphs G, G' within that class and $|V(G)| = |V(G')| = n$. Prove or otherwise justify your response.
   (a) Simple graphs.
   (b) Loopy graphs. (Graphs explicitly containing at least one loop)
   (c) Non-loopy multigraphs. (Contain at least one multi-edge)
   (d) Loopy multigraphs. (Contain at least one loop and one multi-edge)
5. Prove using induction that if some G has no odd cycles then G must be bipartite.
6. G is a connected simple graph. G does not contain a triangle or a path of length 3 as induced subgraphs. Prove that G falls into the isomorphism class of a complete bipartite graph $K_{i,j}$, for some arbitrary i and j.
7. G is a k-regular bipartite graph. Prove that if $k \geq 2$ then G has no cut edge.
8. We have a weakly connected loopless directed graph D with $\forall v \in V(D) : d^+(v) = 1$ and $n = |V(D)| \geq 2$. Answer the following in terms of n, prove or otherwise justify your responses:
   (a) What are the maximum and minimum number of cycles that D can have in terms of n?
   (b) What is the maximum number of cycles if D isn't weakly connected?
   (c) What is the maximum number of cycles if D is neither weakly connected nor loopless?