Homework 2 - Solutions

Friday, February 16, 2024 7:03 PM

① We want to prove that T is connected & acyclic

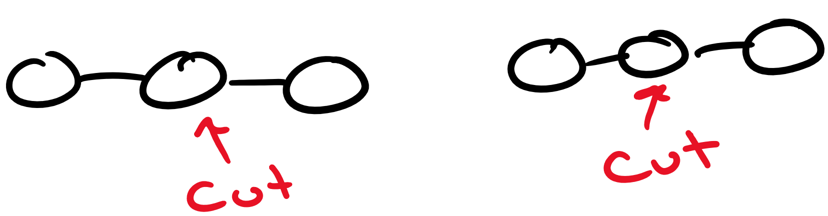
a) Minimally connected => connected

As we've discussed, removing any edge from a cycle will not disconnect the cycle

→ T cannot have any cycles

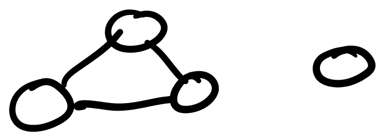
=> T is acyclic □

b) Counter-example



→ a forest also has this property □

c) Counter-example



$|V| = n = 4$

$|E| = m = 3 = n - 1$

→ again, no guarantee of connectedness or acyclicness $\square$

(2) We know $n = m + 1$ in a tree T

→ this implies $\underbrace{\sum_{v \in V(T)} d(v) = 2n - 2 = 2m}$

→ we require at least one non-leaf vertex v, $d(v) > 1$ in a tree with $n > 2$, $m > 1$ ($\sum d(v) = n$ if all leaves)

a) We maximize leaves by only having a single non-leaf vertex as in a star

$$\boxed{n-1}$$

b) We cannot have a single degree-1 vertex, consider a maximal path from a leaf, it must end at a leaf

extremal

→ we require at least two leaves as in a path

$$\boxed{2}$$

$\boxed{c}$

c) consider some $v \in V(T) : d(v) = \Delta(T) > 1$

consider all $u \in N(v)$

extremal

consider maximal paths from each

→ as v is necessarily a cut vertex, these paths are disjoint

→ these paths must all end at a unique degree-1 leaf vertex

⇒ we have $\boxed{\text{at least } \Delta(T) \text{ leaves}}$

Note: to maximize leaves when considering some ΔT, consider the below

We know we have $\sum d(v) = 2m = 2n-2$

$\underbrace{m = n-1}$

→ m is fixed as a function of n

→ This implies that we can maximize l = # leaves for a given n if we maximize the number of edges attached to leaves

Q: How can we do that?

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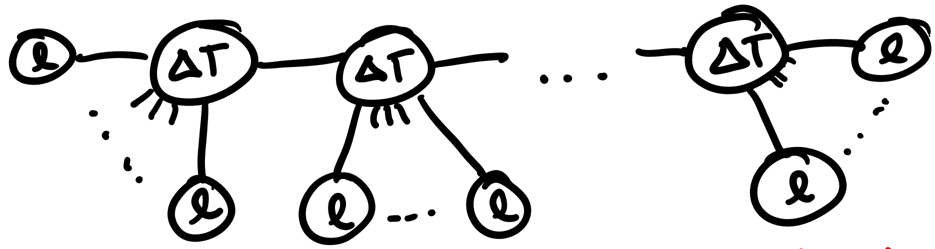
A: Caterpillar graph with vertices of max degree along the path



(feel free to disprove this)

→ We have a tree on n vertices comprised of some number of leaves and the rest are vertices of max degree

Consider our degree-sum formula

$$\sum_{v \in V} d(v) = 2m = 2n-2 = \ell + \Delta(T)\, n_{\Delta(T)}$$

number of leaves; number of max degree verts

$$2n-2 = \ell + \Delta(T)(n-\ell)$$

$$2n-2 = \ell - \ell\Delta(T) + n\Delta(T)$$

$$2n-2-n\Delta(T) = \ell\,(1-\Delta(T))$$

$$\boxed{\Delta(T) \leq \ell \leq \frac{2n-2-n\Delta(T)}{(1-\Delta(T))}}$$

Note: don't need to have solved for this bound

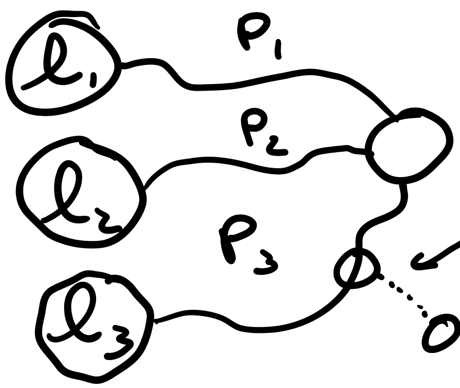
3 a) As we've observed above, the

(2) a) As we've observed above, the only tree with 2 degree-1 vertices is a path graph P_n

$\Rightarrow$ $\boxed{\text{only 1 non-iso. tree}}$

b) We have the basic structure, as an extension of the above

graded as BONUS



Note: more than 1 degree-3 vertex would require more than 3 leaves

So the number of non-isomorphic graphs is the number of unique P_1 P_2 P_3 path lengths using m edges

Note: we can assume that each path length is at least length 1

We are essentially trying to find unique solutions to

$$|P_1| + |P_2| + |P_3| = m$$

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$$\text{where } |p_1|, |p_2|, |p_3| > 0$$

→ The number of ways to distribute m to the three paths can be calculated via the solution to the classic "stars and bars" problem, with known solution

$$\binom{n-1}{k-1} \rightarrow \binom{m-2}{2}$$

However, should we want to find only the unique isomorphism classes, we could use the additional problem constraint of

$$|p_1| \geq |p_2| \geq |p_3|$$

This constraint transforms the problem from "n indistinguishable balls into k distinguishable bins" into "n indistinguishable balls into k indistinguishable bins", which does not have a nice solution does not have a nice solution

→ This is also referred to as the "restricted integer partition" problem, which is well-studied

While there is no nice solution for a given (n, k), it is relatively straightforward to enumerate or calculate for some given values □
(see Google/wiki/etc)

(9) Consider any arbitrary tree T

Iteratively: select a leaf and delete it

→ continue until only a single vertex remains

(basically prüfer code algo)

→ we can reconstruct T by adding back the deleted vertices in the reverse order

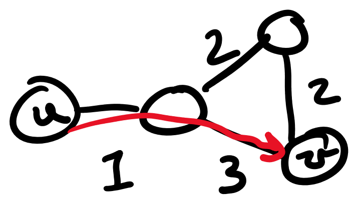
ALGORITHM PROOF (ish)

⇒ as this can be done for any arbitrary T, we can generate all T by iteratively adding leaves □

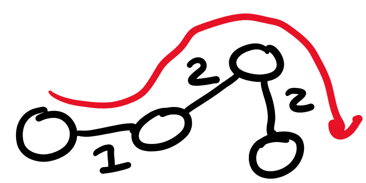
all T by iteratively adding leaves $\square$

(5) EZ proof by "counter-example"

G



MST



$\min_{u,v\text{-paths}} w(u,v\text{-path}) = 4$

$w(u,v\text{-path}) = 5$

(6) Consider T as hypothetical MST on G

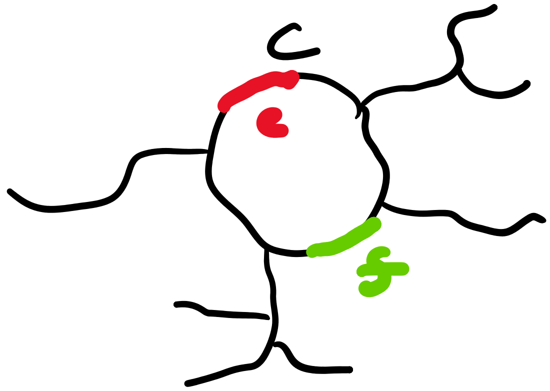
consider cycle C and max weight edge $e \in E(C)$, $e \in E(T)$

We know theres some edge f where $f \in E(C)$, $f \notin E(T)$ and $w(f) < w(e)$

We can construct a new spanning tree T' via $T' = T - e + f$

Is T' still a tree?

Yes → as e is a cut edge on T,



T − e results in two components, adding back f will connect those two components, and as f will be a cut edge, no cycle can result

Is T' still spanning?

Yes → $e = (u, v)$ and u, v still are endpoints on some other edge still in T'

In addition, we know that

$w(T') < w(T)$ as $w(f) < w(e)$

→ This gives us a contradiction on our selection of T

⇒ Hence no MST will have an edge of maximum weight on some cycle C □