Graph Theory Homework 2

Due: 16 Feb 2024 at midnight EST as a PDF on Submitty

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1. Prove or disprove whether each of the below *necessary* properties of a tree T is also *sufficient*. You have no other assumptions about T.
   (a) T is minimally connected (i.e., removing any edge will make T disconnected).
   (b) Every vertex with degree of at least 2 in T is a cut vertex.
   (c) T has $n - 1$ edges.
2. Determine and prove the below bounds on the number of leaves of tree T, where $n \geq 4$.
   (a) What is the maximum number of leaves on an n-vertex tree T?
   (b) What is the minimum number of leaves on an n-vertex tree T?
   (c) Give the tightest provable bound on the number of leaves on tree T relative to $\Delta(T)$.
3. Now consider some enumerative properties of tree T, where $n \geq 4$.
   (a) How many non-isomorphic n-vertex trees exist that have 2 leaves?
   (b) How many non-isomorphic n-vertex trees exist that have 3 leaves?
4. As stated in class, weak induction is particularly applicable to trees, because we can generate all possible tree configurations by iteratively adding a single new leaf to an existing tree. Prove this. (Note: using weak induction here might result in circular logic, so be careful.)
5. Consider the minimum-weight spanning tree T of some weighted simple connected graph G and some arbitrary $u, v \in V(G)$. Prove or disprove: a u, v-path in T has the minimum sum of edge weights of all possible u, v-paths in G.
6. Consider some cycle C in a connected weighted graph G. Let $e \in C$ be the edge of maximum weight in C, where $\forall f \in C, f \neq e : W(e) > W(f)$. Prove or disprove whether there exists a minimum weight spanning tree T of G that contains e.