Assume M_1 and M_2 as 2-distinct perfect matches on tree T

Consider symmertic difference as $F = M_1 \Delta M_2$

→ We know leaves must be matched to their sole neighbor, so no edges of leaves are in F

Define Q = set of leaves and their sole neighbors

Consider $T' = T - Q$

→ All components of T' will have a P.M. using M_1 or M_2, and all leaves in T' will be matched to their sole neighbor

→ Hence, no edges of leaves of T' will have an edge in F

→ we repeat this process until

→ we repeat this process until no edges remain, and observe that F will remain empty,

⇒ Hence, M_1 and M_2 must be equal, so any perfect match on a tree is unique □

② Assume we have some maximal match M' where $|M'| < \frac{|M|}{2}$

→ $\forall e = (u, v) \in M'$, there is at least 1 and at most 2 edges in M that are incident to u or v

→ This implies at most $2|M'|$ edges in $|M|$ are incident to some $v \in V(M')$

→ From our intial assumption we have $|M'| < \frac{|M|}{2} \rightarrow 2|M'| < |M|$

Together, this implies that at

Together, this implies that at most $2|M'|$ are saturated by an edge in $|M|$, but the strict inequality implies there must be at least one edge in M not incident to a vertex in M'

$\Rightarrow$ As this edge can be added to M', our choice of M' is a contradiction, so we must have $|M'| \geq \frac{|M|}{2}$

(3) Define: C = vertex cover

$\bar{C}$ = complement of cover

Note: every edge of G is incident on at least one $v \in C$

$\hookrightarrow$ so every edge is therefore incident on <u>at most</u> one vertex in $\bar{C}$

$\Rightarrow$ so there is no edge between

$\Rightarrow$ so there is no edge between any $u, v \in \overline{C}$, so $\overline{C}$ is an independent set $\square$

(4) Recall Tutte's: $\forall S \subseteq V(G) : o(G-S) \leq |S|$

consider some S where $|S| \geq 1$
(condition trivially holds for $|S| = 0$)

Consider $G' = G - S$
and odd component H of G'

Note: Edges connecting S to H is bounded below by $k - 1 \leq \overset{\text{edges from } S \text{ to } H}{x}$

Note 2: Degree sum of H

$$\sum_{v \in V(H)} d(v) = k\,\overset{\text{odd}}{|V(H)|} - x = 2|E(H)|$$

% parity %

$\hookrightarrow$ implies k, x = both even or both odd

AND since $k - 1 \leq x$ we also will have $k \leq x$ (due to parity)

We repeat this for all possible H_i

We repeat this for all possible H_i

→ so at least $k * o(G-S)$ edges to S from all of H_i

→ and a degree sum of S

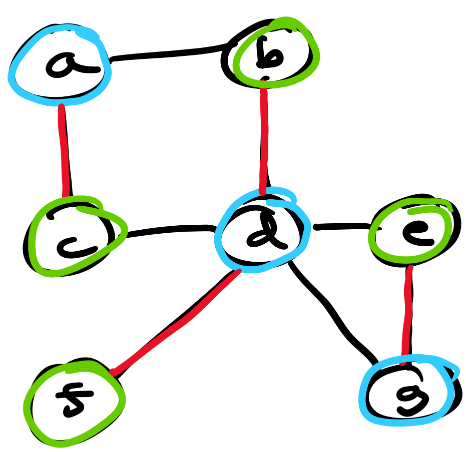
$$\sum_{v \in S} d(v) = k|S|$$

$$\Rightarrow \quad k * o(G-S) \le k|S|$$

$$o(G-S) \le |S|$$

which holds for all possible choices of S and k □

5



a) | = edge cover

○ = vertex cover

Note: doesn't need to be minimum, can even just be $E(G)$, $V(G)$

b) No, define $S = \{d\}$

$$o(G-S) = 2 > |S| = 1$$

so Tutte's doesn't hold □

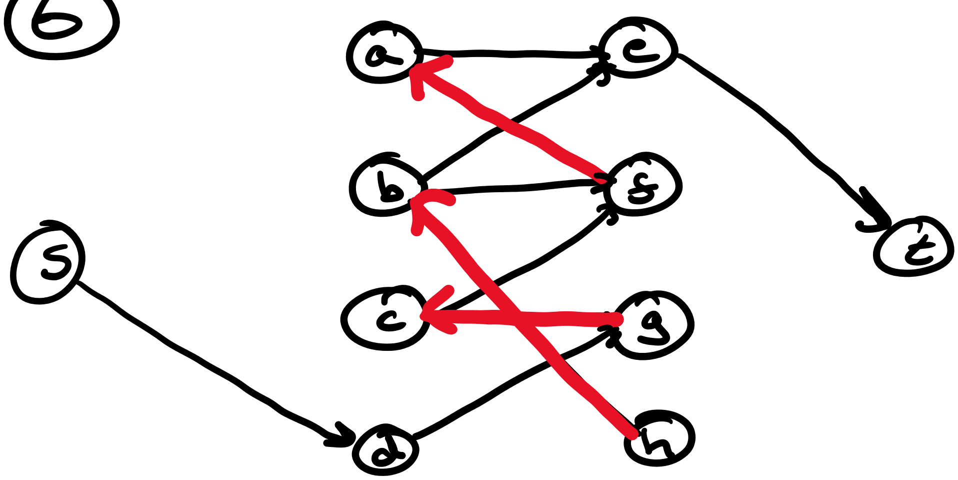
so Tutte's doesn't hold □

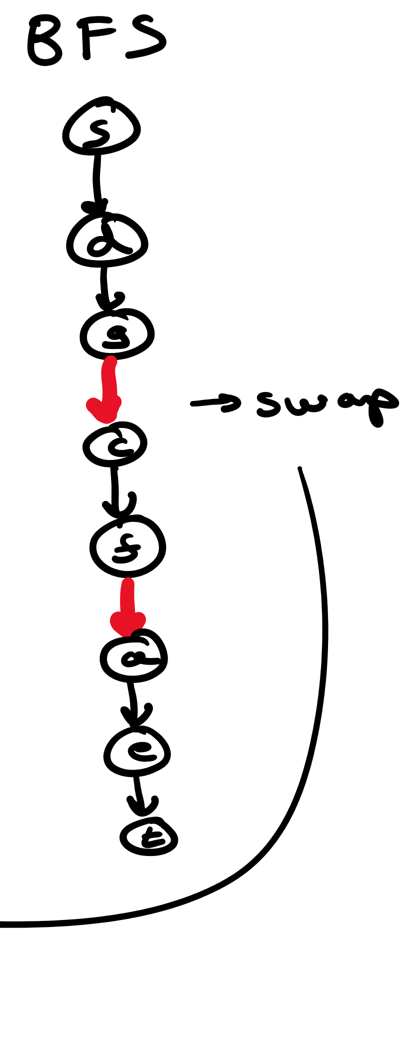
c) $O = \bar{C}$ → independent set

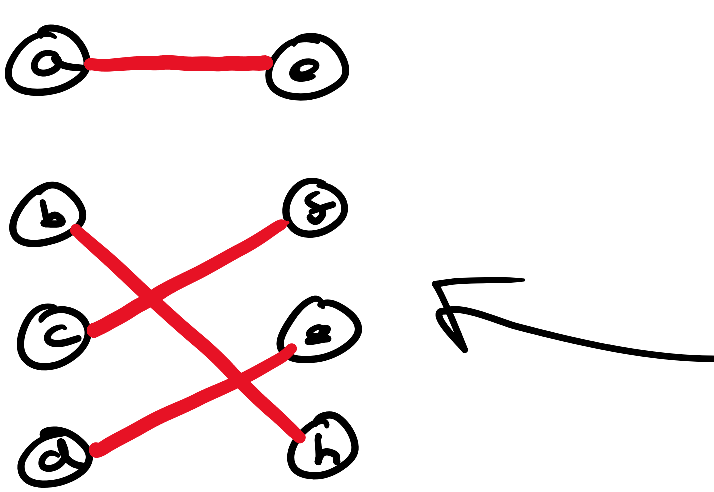
We note $\forall e \in E(G)$, at most one endpoint of e is in $\bar{C}$

→ $\bar{C}$ is independent set □

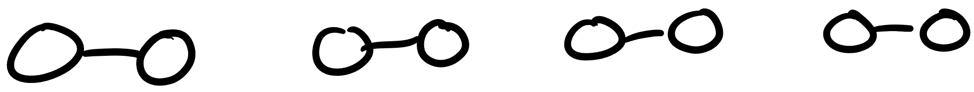
6





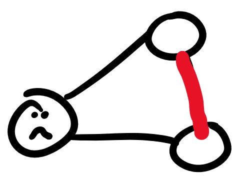


7 a) The only 1-regular graph is comprised solely of K_2 components



→ These will alway trivially have a P.M. □

b) Any odd cycle will do



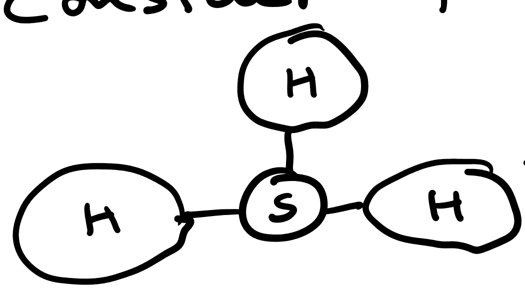
c) Get's a little trickier

Note: by DSF (degree sum formula), $|V(G)|$ = even

so there won't be a trivial counter-example

One approach is just draw all possible 3-regular configurations and brute force an answer

OR: consider Tutte's

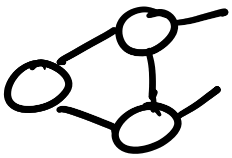


→ does some H exist where H is 3-regular except for a degree-2 vertex?

for a degree-2 vertex?

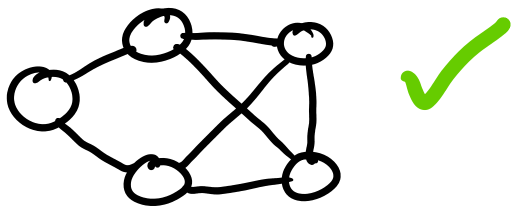
Note: $|V(H)|$ must be odd

$|V(H)| = 3?$

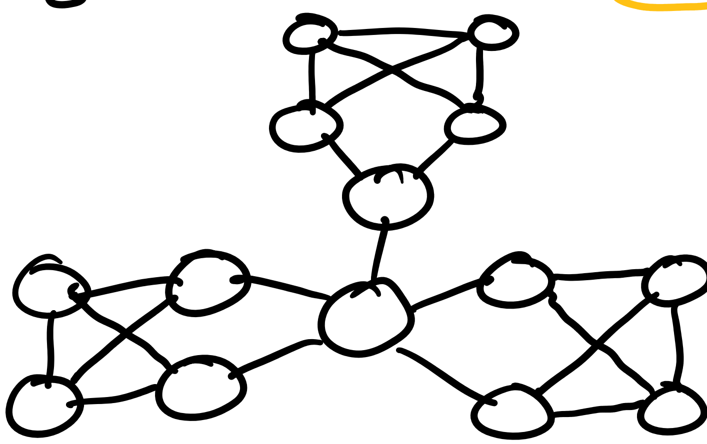


Nope

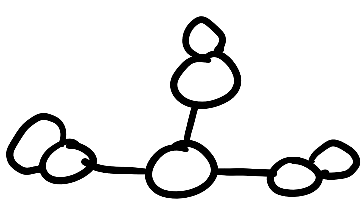
$|V(H)| = 5?$



All together now...



However: I never required that your example is simple



so this would also work