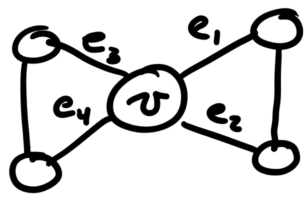
Homework 4 - Solutions

Saturday, March 30, 2024 4:15 PM

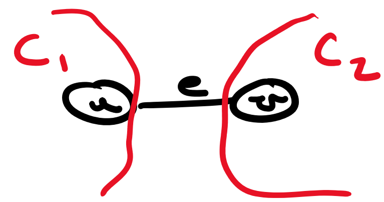
① EZ disproof via **EXAMPLE**



v is a cut vertex

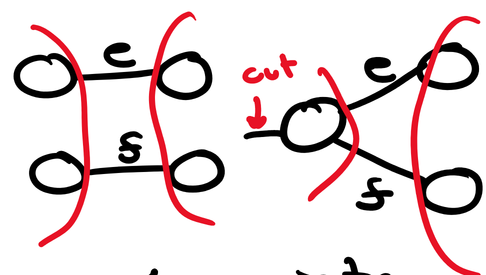
none of $e_i \in N(v)$ are cut edges

② Consider if $\kappa'(G) = 1$



→ either neighbor of a cut edge is a cut vertex, so $\kappa(G) = \kappa'(G) = 1$

Consider if $\kappa'(G) = 2$



→ no vertex cut of size 1 exists, since as above it would imply a single edge cut

$\Rightarrow$ so $\kappa(G) = \kappa'(G) = 2$

Consider if $\kappa'(G) = 3$

→ all $v \in V(G)$ must have $d(v) = 3$

→ $\exists\, u,v$-edps $\forall u,v \in V(G)$
edge disjoint paths

→ no vertex can be on 2 separate

→ no vertex can be on 2 separate edps, hence these paths are all vertex disjoint

$\Rightarrow$ By Menger: $K(G) = K'(G) = 3$ □

(3) if $K(G) = 0$ then $K'(G) = 0$

if $K'(G) = 1$ 

→ for cut e, some endpoint v is a cut

$\Rightarrow$ $K'(G) = K(G) = 1$ (deleting v deletes e)

if $K'(G) = 2$, $K'(G) = 3$

→ By Menger, we have two / three edps between all $u, v \in V(G)$

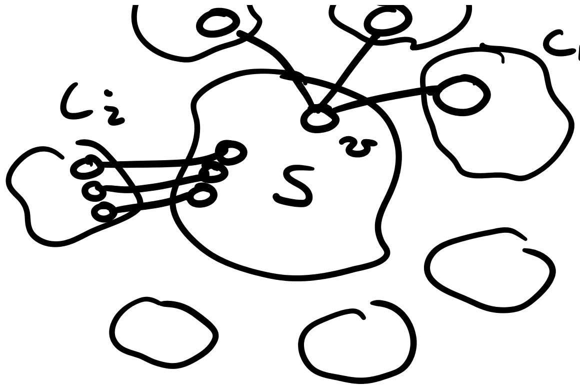
→ as any internal vertex can only be on one edp, these paths are also vertex disjoint

$\Rightarrow$ $\boxed{K(G) = K'(G)}$ □

(4)

Consider some $S \subseteq V(G)$ and Tutte's

1) Any $v \in S$ can connect to



1) Any $v \in S$ can connect to at most k components of $G-S$

2) Each component C_i has at least k edges to S

Consider bounds on cut E of $G-S$

$|E| \leq Sk$ by 1) above

$|E| \geq Nk$ by 2) above, where N is # components of $G-S$

$o(G-S) \leq N$, trivially
odd comps

so $Nk \leq |E| \leq Sk$

and $o(G-S) \leq N \leq S$

$\Rightarrow o(G-S) \leq S$ for all choices of S, so G has P.M. per Tutte's $\square$

(5) Consider cuts $E = [S, \bar{S}]$, $F = [T, \bar{T}]$

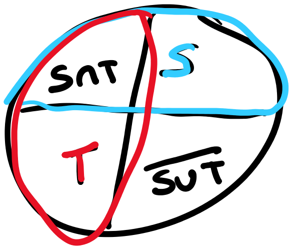
Consider cuts $E = [S, \bar{S}]$, $F = [T, \bar{T}]$

and symmetric difference

$$G = E \Delta F$$

or union $\cup$

$$= [S \cup T - S \cap T, \overline{S \cup T} + S \cap T]$$



Q: is this also a cut?

→ If $S \cap T = \emptyset$, trivially yes

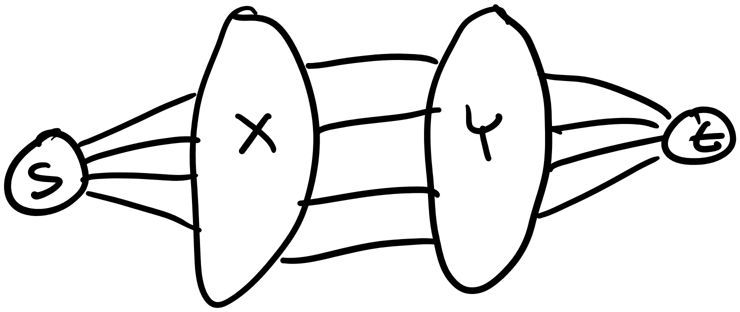
So assume $\exists v \in S \cap T$

→ To be a valid cut, v must not be in $\overline{S \cup T}$, or $v \notin \bar{S}, v \notin \bar{T}$

⇒ we observe this trivially, as v per our assumptions is in both of S and T

⇒ Hence $G = [S \cup T - S \cap T, \overline{S \cup T} + S \cap T]$ is also a valid cut □

⑥ Consider a setup similar to our matching algo:



$\forall x \in X: (s, x) \in E(B')$ $\quad$ $\forall y \in Y: (y, t) \in E(B')$

→ we note that the max # idps gives an equivalent set of a match on B using (x, y) edges (max, as a larger match would give more idps)

→ Per Menger's theorem, we also have a vertex cut of equivalent size

→ as removing this cut fully disconnects X and Y (or s & t), it covers all (x, y) edges as a vertex cover

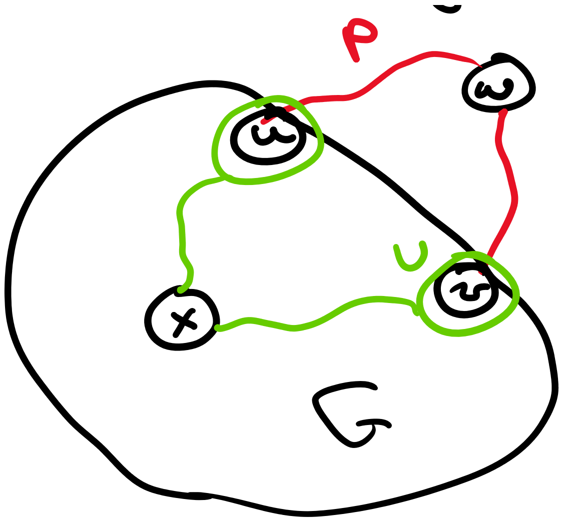
(and a smaller cover would give a smaller cut, contradicting the max idps we originally assume)

⇒ This gives max match = min cover □

(7) 

we have 2-connected G

(7)



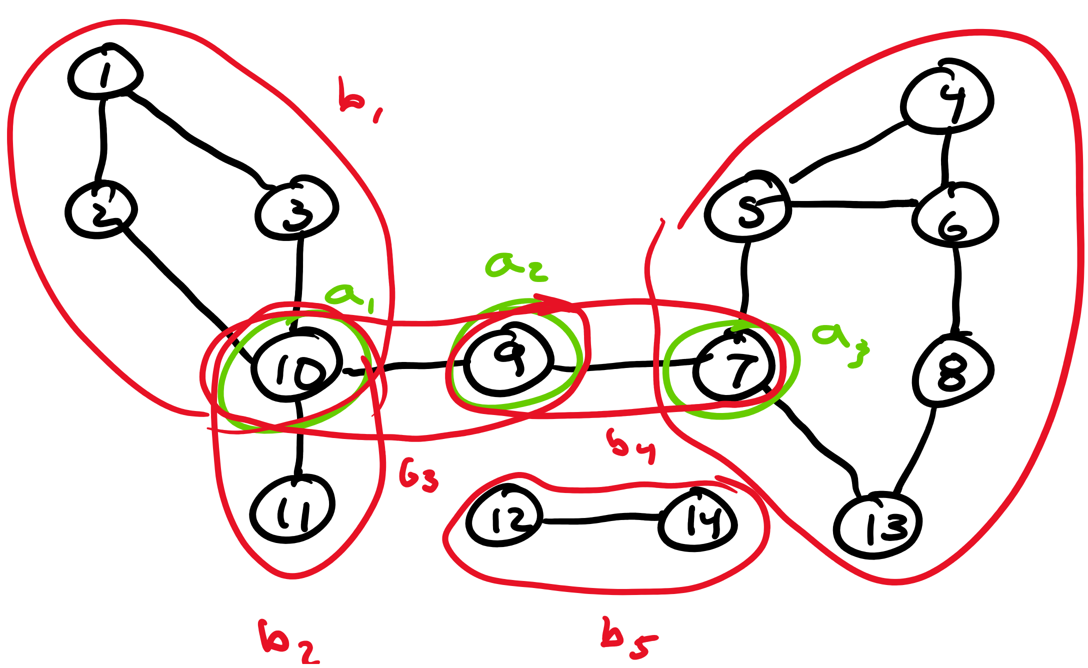
we have 2-connected G and u,v-path P

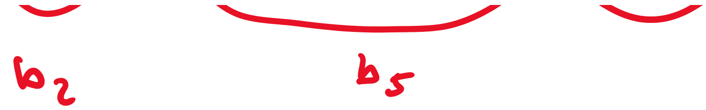
→ In class, we showed $\exists\, x, U$-fan of size k for any $U \subseteq V(G) : |U| = k$ in a k-connected G

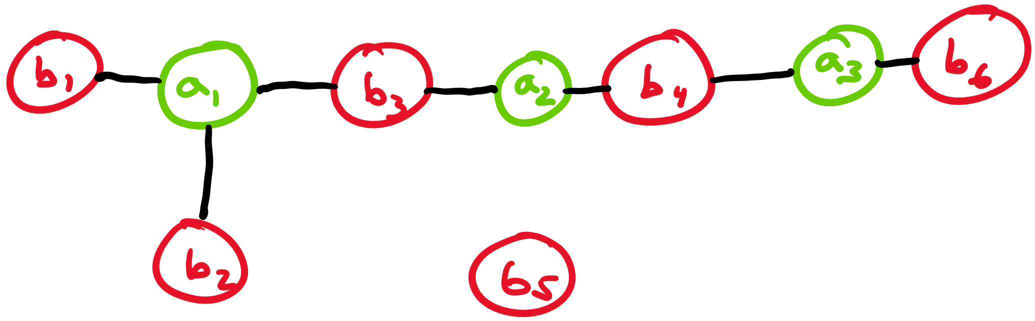
→ as G is 2-connected, set $U = \{u, v\}$ and we have an x,u-idp and x,v-idp that are disjoint from each other

⇒ combined with P, we trivially have 2 x,w-idps □

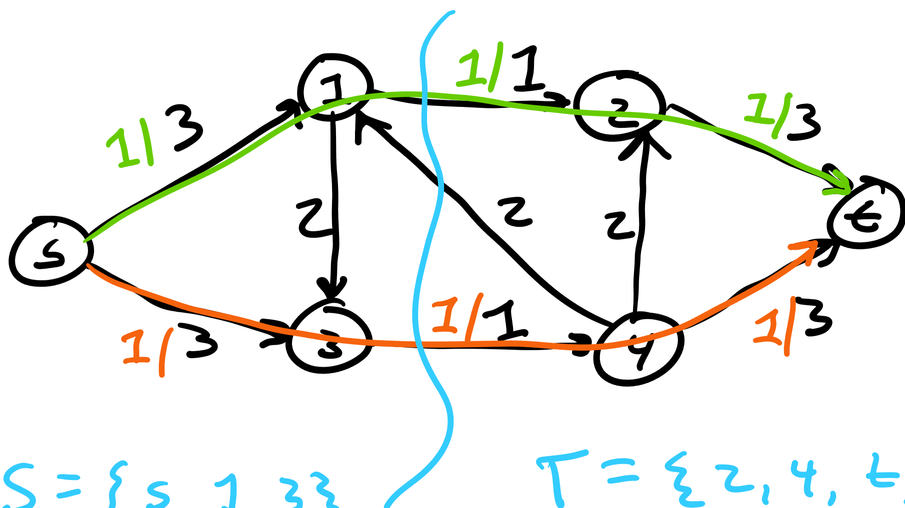
(8)







(9)

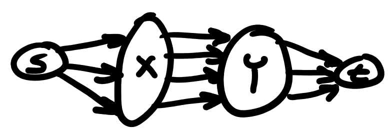


max flow = 2

$S = \{s, 1, 3\}$ $T = \{2, 4, t\}$

$$|[S, T]| = 2 = \text{max flow} \quad \square$$

(10) Similar to problem 6



- add vertices s, t : $\forall x \in X$: (s, x) directed
  : $\forall y \in Y$: (y, t)
- orient (x, y) edges from x to y
- give all edges unit capacity

- solve for max flow

→ we get a matching on (X,Y) edges with unit flow, as each vertex can have at most one unit of flow through it, this is max, as a larger match gives larger flow

→ we also get a min edge cut, which is also a min vertex cut, as we have each flow unit forming an idp, so this follows via Menger

⇒ Hence, this vertex cut will also be a minimum cover, as it is incident to a minimum edge cut that separates X, Y (smaller cover implies smaller cut)

⇒ so min cover = max match □