Homework 5 - Solutions

Wednesday, April 3, 2024 5:25 PM

① Consider an optimal coloring on G

Order colors in any arbitrary way

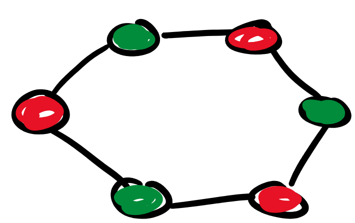
Order vertices with a given color in any arbitrary way

→ apply greedy coloring

=> as vertices in a color set are all independent, the maximum color they will get assigned is only a function of how many prior color sets were processed

=> this gives an optimal ordering (and coloring) □

② $\exists Z : C_6$



$\Delta(C_6) = 2 = \chi(C_6)$

③ Assume $\Delta(G) \geq 2\sqrt{n}$, otherwise trivial (Brooks)

Note: every neighborhood forms an

Note: every neighboorhood forms an independent set

Case 1: we have fewer than $\sqrt{n}$ verts v wit a degree $d(v) \geq 2\sqrt{n}$

→ we color all $N(v)$ with at most $\sqrt{n}$ colors, need at most $\sqrt{n}$ more colors for the remaining per Brooks ⇒ $\#\text{colors} \leq 2\sqrt{n}$

Case 2: we have more than $\sqrt{n}$ such v

→ Iteratively color $N(v)$ with most remaining uncolored

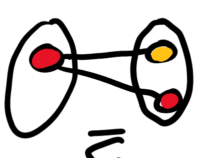
→ after $\sqrt{n}$ iterations we have no more such v → Brooks → $\#\text{col} \leq 2\sqrt{n}$ □

④ $\omega(G)$ = size of largest $\underbrace{\text{ind. set}}_{\text{becomes clique in } G}$ in $\overline{G}$

$= |V(G)| - (\text{min vertex cover on } \overline{G})$

→ complement of ind. set

$\chi(G)$ = min number of edges to cover all of $|V(\overline{G})|$



$\overline{G}$

if verts share edge they can

$= |V(G)| - (\text{max match})$

(3.1.22 in the book)

if verts share edge they can be colored same in G

(3.1.22 in the book)

so $K-E \rightarrow \omega(G) = \chi(G)$ □

5 We know G has a SEO

Consider this SEO in reverse

→ when v_i is added to G_i
$N(v_i)$ is a clique

→ we can greedily color v_i
with color $|N(v_i)|+1$

→ eventually, we'll have some v_j
in largest clique K_n getting
color $n = \omega(G) = \chi(G)$

Note: there is no way to induce
a subgraph H on G s.t. H
has a chordless cycle
(any chord will always induce)

⇒ hence the above applies
to any $H \subseteq G$ thus $\chi(H) = \omega(H)$

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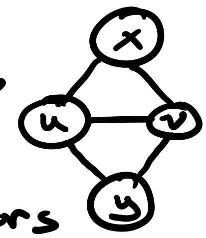
and G is perfect $\square$

(6) We know $|E(S)| \neq 6$, as that would imply S is K_4, and deleting any edge not in S would not affect $\chi(G) = 4$

Consider $|E(S)| = 5 \rightarrow$

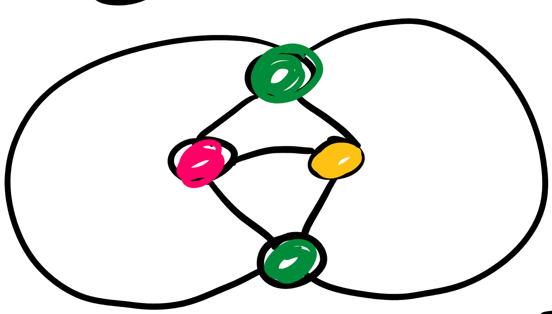


only configuration

→ u, v have different colors

→ x, y have same or different colors

Consider an S-lobe



↳ must be 3-colorable, so x, y have same color

→ this applies to all S-lobes

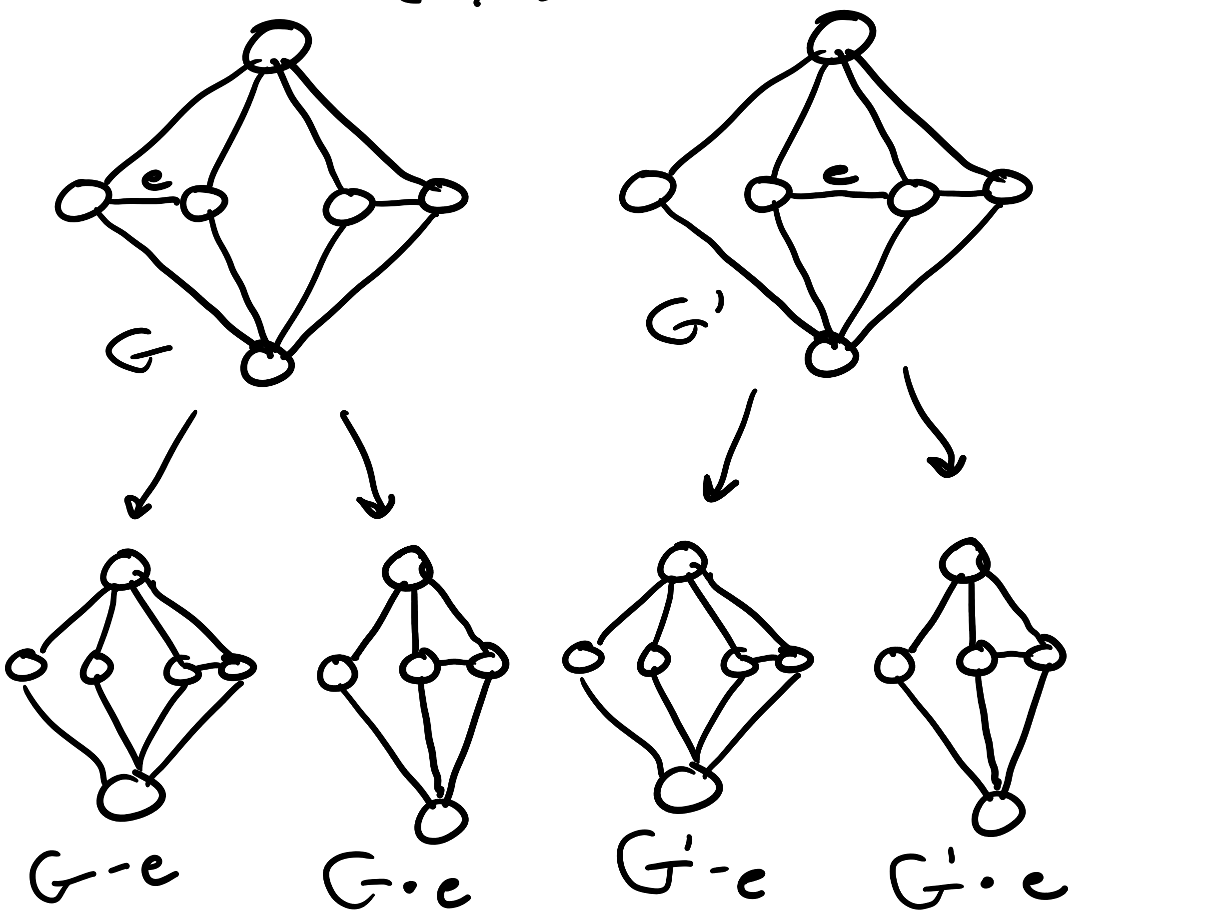
=> there exists a valid 3-coloring of S and all S-lobes of S and all S-lobes

contradiction

So S must have $|E(S)| \le 4$ $\square$

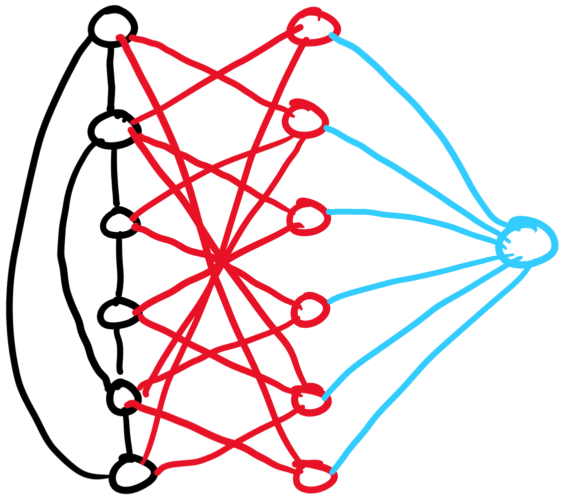
7. Consider a single step of the recurrence for each

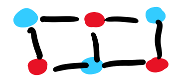
$\chi(G,k) = \chi(G-e,k) - \chi(G\cdot e,k)$



$\Rightarrow$ so they both have the same chromatic polynomial as they chromatic polynomial as they have the same result from a step of the recurrence □

⑧

a) 

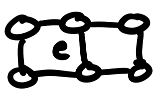
$\chi(G) =$  $= 2$

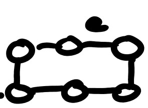
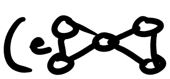
$\chi(G') = 2+1 = 3$

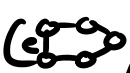
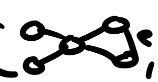
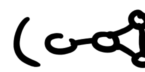
$\omega(G) = 2$

$\omega(G') = 2$

via our proof of Mycielski

b) $\chi($  $, k)$

$\chi($  $, k) - \chi($  $, k)$

$\chi(T_5, k) - \chi($  $, k) - \chi($  $, k) + \chi($  $, k)$

(tree on 5 verts)

$\cancel{\chi(T_5, k)} - \cancel{\chi(T_4, k)} + \chi(C_4, k) - \cancel{\chi(T_5, k)} + \cancel{\chi(T_4, k)}$

$+ \chi(T_4, k) - \chi(T_3, k)$

$= \chi(C_4, k) + \chi(T_4, k) - \chi(T_3, k)$

(from class)

$= k(k-1) + 2k(k-1)(k-2) + k(k-1)(k-2)(k-3)$

$+ k(k-1)^3 - k(k-1)^2$

for $k = 1 \rightarrow 0 + 0 + 0 + 0 - 0$

$k = 2 \rightarrow 2(1) + 0 + 0 + 2 - 2 = 2$

$\Rightarrow$ so $\chi(G) = 2$

c) No $\rightarrow$ G has two chordless cycles so G is not chordal and therefor has no SEO