Homework 6 - Solutions

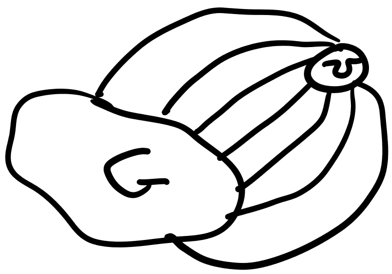
Sunday, April 14, 2024 12:32 PM

(1) consider outerplanar G

Construct $G' \rightarrow V(G') = V(G) + \{v\}$

$E(G') = E(G)$

$+ \{(u, v)\ \forall u \in V(G)\}$



→ We note G' must also be planar, as we can trivially draw edges from v to all u in $V(G)$

→ Hence G' is 4-colorable

We also note that v would need a unique (4th) color, as its adjacent to all other u

→ G is 3-colorable □

(2) G has G' subdivision

$\iff$

G has some $H \subseteq G$ that can

G has some $H \subseteq G$ that can produce G' from edge contractions

($\Rightarrow$) trivial, just contract edges along subdivision to get G'

($\Leftarrow$) Consider some minimal H

Consider some $e \in E(H)$ and $H \cdot e$
$e = (u, v)$

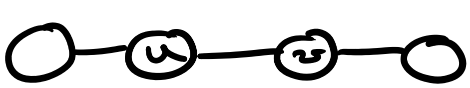
We need to show that as we expand edges from $G' \rightarrow H$ that we retain a G' subdivision

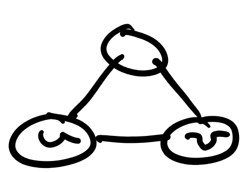
Case 1: $d(u)$ or $d(v) = 1$

$\rightarrow$ vertex deletion, which is contradictory to our minimal H

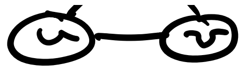
Case 2: $d(u)$ and $d(v) = 2$

$\rightarrow$ u and v are either on a subdivided path or K_3



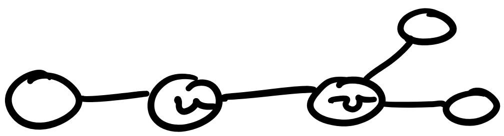


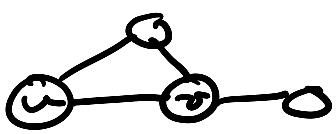
$\rightarrow$ if K_3, our selection



→ if K_3, our selection of H wouldn't have this unless G' is C_2

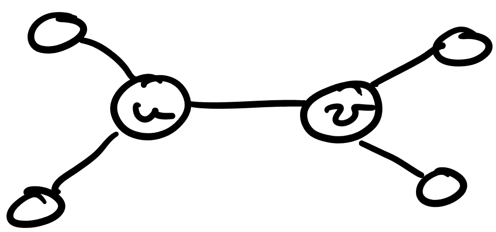
Case 3: $d(u)=2$, $d(v)=3$



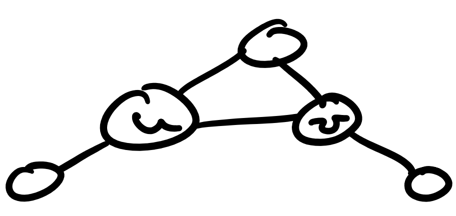


→ Same as above, if they are in K_3 then there is some $C_2 \subseteq G'$

Case 4: $d(u)=3$, $d(v)=3$



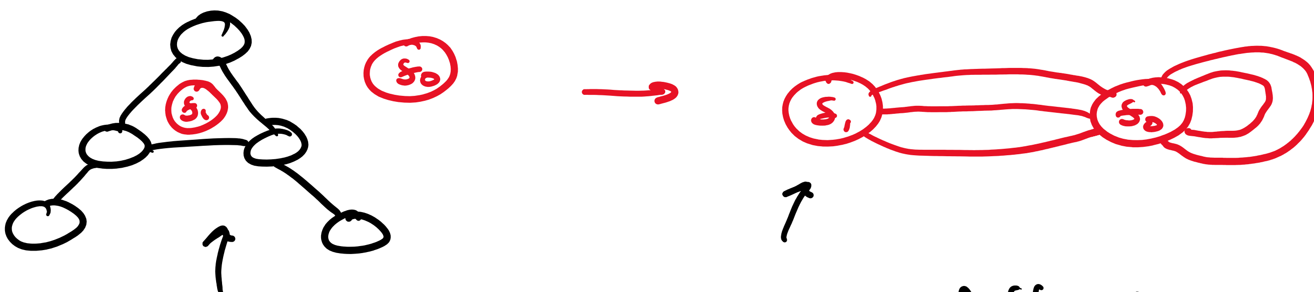
→ If they share no neighbors, then contracted vertex has degree of 4, contradicting assumptions on G'



→ Hence they share a neighbor and fall into similar cases as above

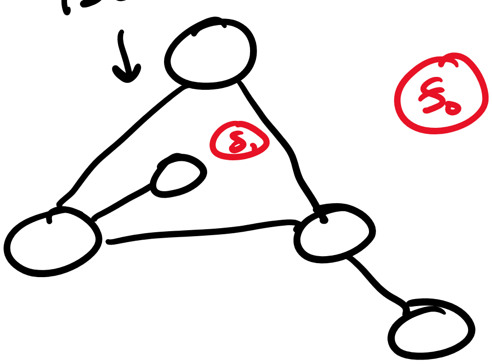
⇒ As lengthening of subdivided path or cycle is only possibility, we retain the G' subdivision □

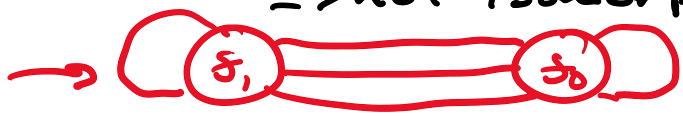
③ EZ Eggzamples



isomorphic

Note: differing self loops on faces ⇒ not isomorphic □





④ To maximize edges → all faces are of length 3, except for the outer face of length $n = |V(G)|$

length $n = |V(G)|$

Consider sum of face lengths

$$2\overset{|E(G)|}{e} = \sum \ell(f_i)$$

$$= 3(f-1) + n$$

\# of faces

$$2e = 3f - 3 + n$$

We'll solve in terms of f

then  and  in Euler's

$$2e + 3 - n = 3f$$

$$\frac{2e+3-n}{3} = f$$

$$n - e + f = 2$$

$$\left(n - e + \frac{2e+3-n}{3} = 2\right) * 3$$

$$3n - 3e + 2e + 3 - n = 6$$

$$2n - e + 3 = 6$$

Solve for e

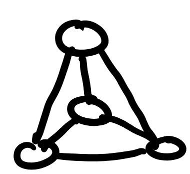
$$\boxed{e = 2n - 3}$$

max edges in outer planar G □

5 Pretty much proof 7 from our Kuratowski theorem proof

Consider some simple planar G (and a triangulation containing G)

Induction on $|V(G)|$

Basis: $P(4) = K_4 \rightarrow$  all lines straight (ish)

$G = P(n)$ case

$P(k) = P(n) \cdot e$

we know edge contraction can't create a K.S., so $P(k)$ is planar

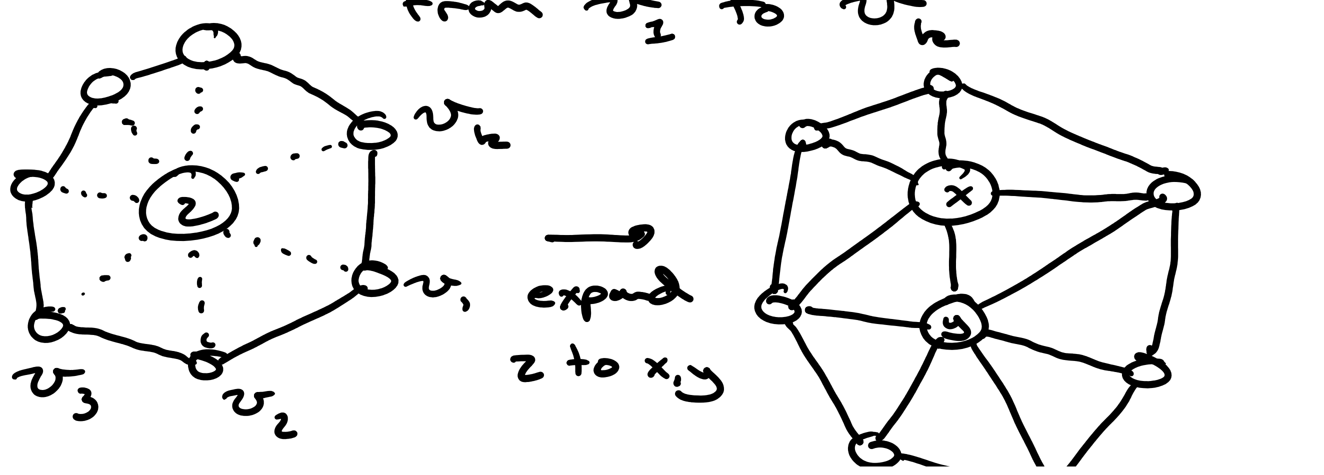
(we can also ignore multi-edges as they are trivial for planarity)

I.H. on $P(k)$ gives us a straight-line embedding

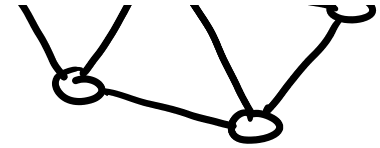
Bring it back to $P(n)$

$z \leftarrow (u,v) = e$, contracted edge

All $N(z)$ form a face around z, we order these neighbors from v_1 to v_k



v_3 v_2 ... x,y



→ we know from (7) that x,y share at most 2 neighbors where $N(x)$ is a subset of $v_2 \dots v_j$

→ we place x,y after expansion such that x has straight line edges drawn to this subset (possible since z did)

→ y will have straight line edges to the rest (also possible size z did)

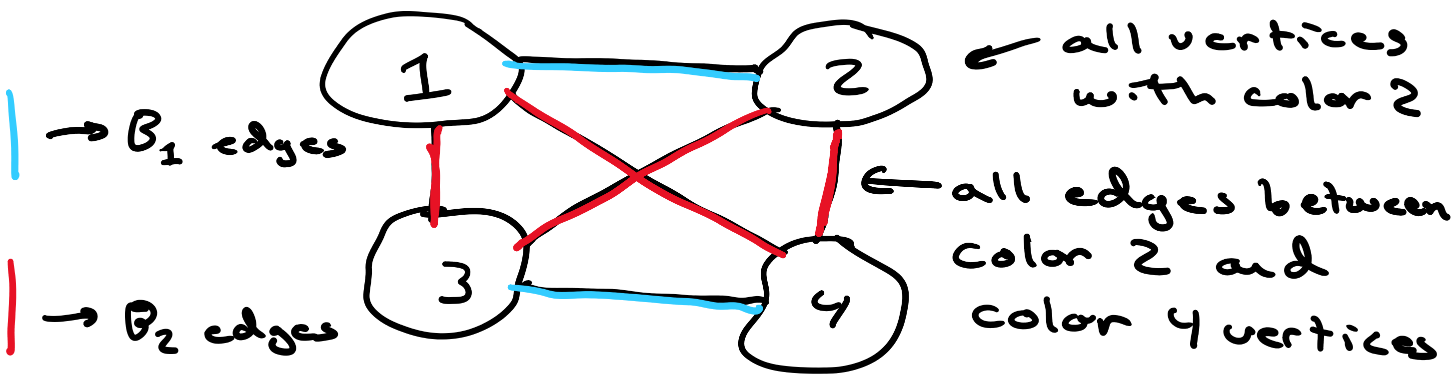
⇒ we have a straight line embedding of G □

(6) Consider 4 colors $\{1, 2, 3, 4\}$

we wish to create a decomp

containing 2 bipartite graphs

Note the graph can be drawn in terms of color sets



We construct two bigraph B_1, B_2 as follows

$B_1 \to$ gets all edges between colors $\{1,2\}$ and $\{3,4\}$ with bipartite sets $\{1,3\}, \{2,4\}$

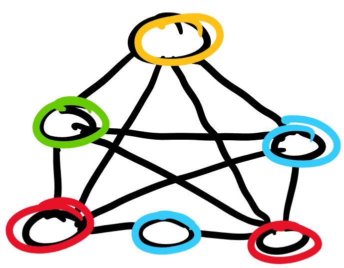
$B_2 \to$ gets all remaining edges with bipartite sets $\{1,2\}, \{3,4\}$

→ As this construction avoids adding edges with a set it results a edges with a set, it results in two bipartite graphs

→ since it uses all edges it is a decomposition □

⑦ a) counter-example

K_5 with subdivided edge

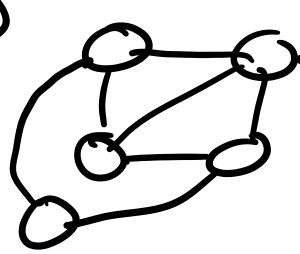


→ 4-colorable but nonplanar

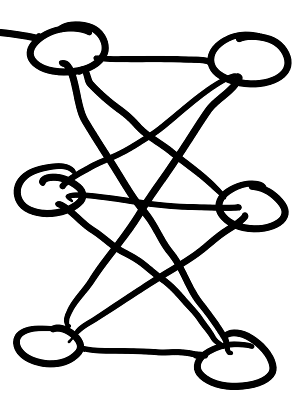
b) counter-example as above

→ Has no K_5, but still has a K_5 subdivision

c)



K_4-sub



$K_{3,3}$

Counter-example can have a $K_{3,3}$(sub)

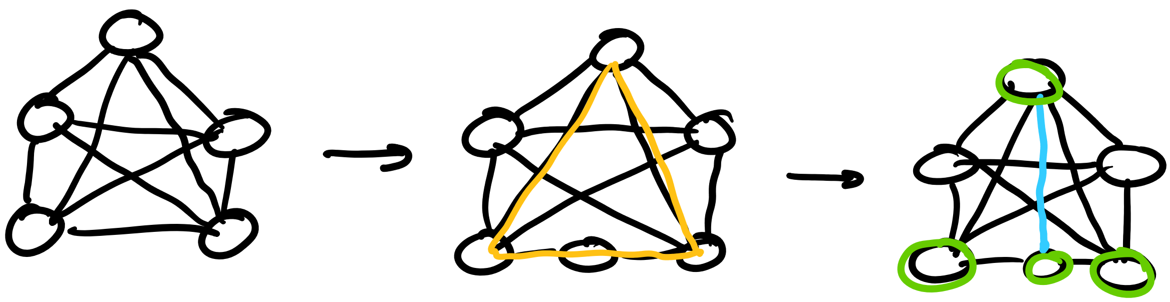
$K_{3,3}$

d) Can G contain a K_5 or $K_{3,3}$ subdivision

Consider K_5:

→ G can't have K_5 since the biggest clique can be K_3

What about a subdivision?

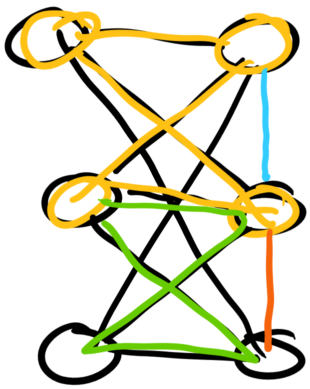


Introduces a chordless cycle C, we must add chord f, which creates K_4 induced on green labeled vertices

Note: other choices of f will also result in K_4

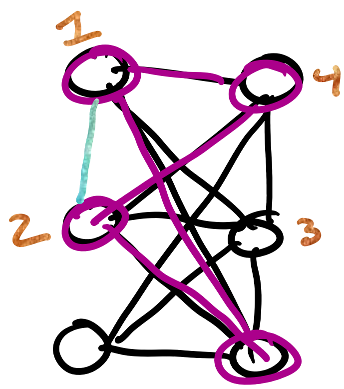
will also result in K_4
in order to make cycles
not chordless

Now consider $K_{3,3}$:



we have chordless cycle C_4, we add chord f

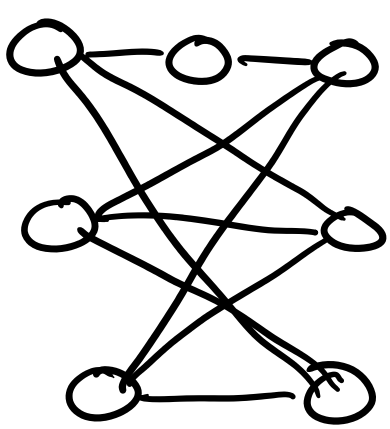
C_4 needs e



C_4 needs g

→ we have K_4 on {1, 2, 3, 4}

And finally a $K_{3,3}$ subdivision



→ we not theres a chordless cycle, repeating the above logic will resut in a K_4

a K_4

$\Rightarrow$ so G cannot contain a K_5 or $K_{3,3}$ subdivision and therefor must be planar $\square$