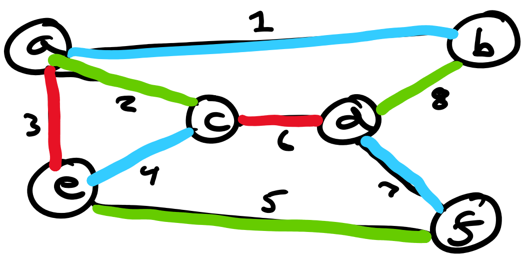
Homework 7 - Solutions

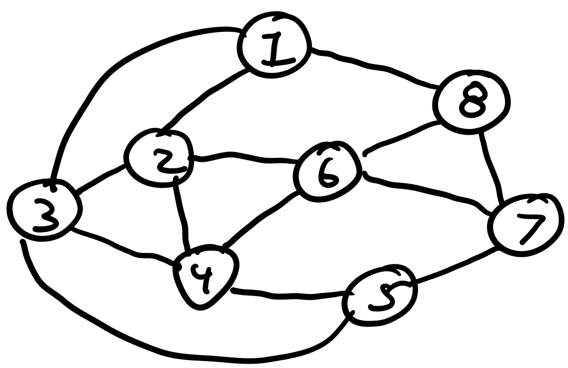
Saturday, April 20, 2024 8:02 PM

① a)

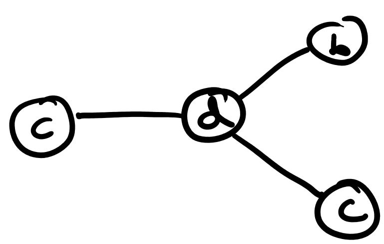


$\chi'(G) = 3$

b) using above edge labels



c) No → G has a claw

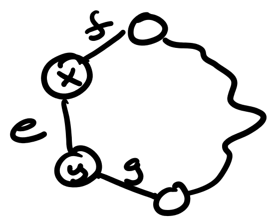


$\Rightarrow$ so as proven in class, there is no such H □

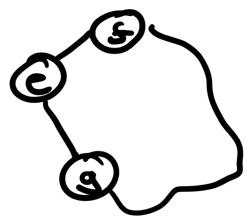
② $G \cong L(G) \iff G$ is 2-regular

($\Leftarrow$) We know G must be a cycle

($\Leftarrow$) We know G must be a cycle



$\rightarrow$ each edge $e \in V(G)$ is incident with exactly two other edges f, g on separate endpoints



$\rightarrow$ this becomes a path in $L(G)$, we can repeat this logic around G in one direction

$\rightarrow$ we will eventually complete a cycle of length $|E(G)|$

$\Rightarrow$ as $|V(G)| = |E(G)|$, this cycle is the same length as G and is isomorphic

($\Rightarrow$) Assume $G \cong L(G)$ (and connected)

$\rightarrow |E(G)| = |V(L(G))|$

$\rightarrow |V(G)| = |E(L(G))|$

as $|V(G)| = |V(L(G))|$

$\rightarrow |E(G)| = |V(G)|$

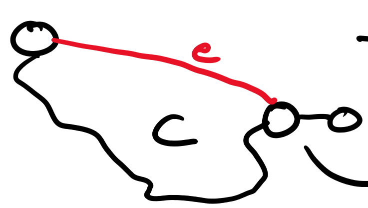
so G has exactly 1 cycle

so G has exactly 1 cycle
(and thus so does $L(G)$)

→ So G is a tree with one additional edge

Note: this edge cannot connect two degree ≥ 2 vertices, as that would result in at least 2 cycles via 2 cliques of $K_{n\geq 3}$

Q: can it connect a leaf to a non-leaf vertex?



→ cycle C becomes a cycle in $L(G)$

→ we have another clique, which add a cycle ✗

Note 2: the above logic also implies that there is no other degree ≥ 3 vertex in the tree, as that would give another cycle in $L(G)$

→ G and L(G) are path graphs with an additional edge connecting the leaves

⇒ they are cycles and therefor 2-regular □

3. We proved in class for bipartite graphs that $\Delta(G) = \chi'(G)$

→ Biclique $K_{i,j}$ is bipartite

$\Rightarrow \Delta(K_{i,j}) = \chi'(K_{i,j})$ □

See lecture 22 / book for proof

4. We will construct G' as a $k = \Delta(G)$-regular simple graph containg G

PROOF BY **CONSTRUCTION**

→ add vertices to smaller of bipartite sets until equal in size

→ While $\exists x \in X, \exists y \in Y : d(x) < k, d(y) < k$
add edge (x, y) to G'

→ as degree sums must be equal, we can continue to find (x,y) pairs until G' is k-regular

(there can be no $x : d(x) < k$ without some $y : d(y) < k$)

=> this gives us our $k = \Delta(G)$-regular G' $\square$

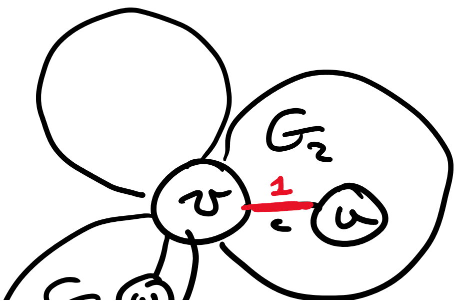
5 G must have $|V(G)| =$ even

→ if $\chi'(G) = \Delta(G)$ then each color must form a perfect match

$G - v$ has some odd component

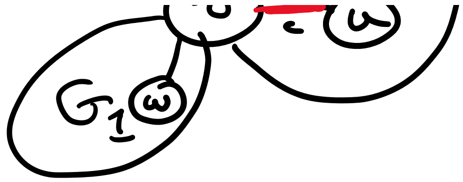
→ $|V(G-v)| =$ odd

Consider cut vertex v and odd comp G_1 and some even comp G_2



→ $\exists u \in G_2$ where $e = (u,v)$ and e has color 1

→ since color 1 is



→ Since color 1 is not incident on v from some $w \in V(G_1)$, all vertices in G_1 must have that color incident in G_1

→ so color 1 must form a P.M. on G_1

Contradiction

↳ against our choice of odd G_1, which can't have a P.M.

$\Rightarrow \chi'(G) = \Delta(G) + 1$ $\square$

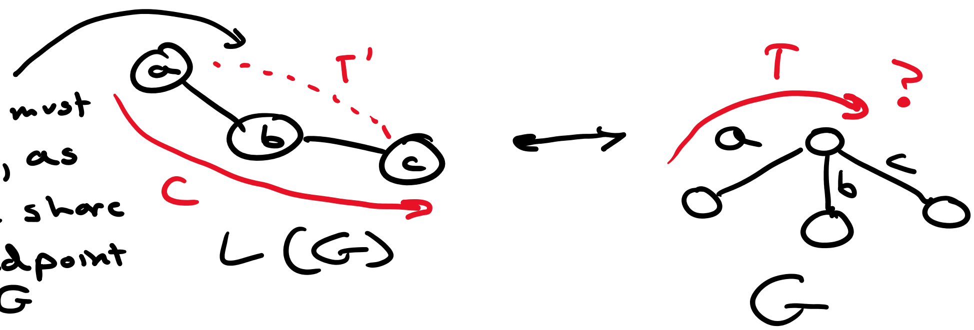
6 $L(G)$ Hamiltonian $\Leftrightarrow$ G has closed trail visiting at least one endpoint of each edge

($\Rightarrow$) We'll construct a closed trail T

→ The vertices of $L(G)$ correspond to a visitation order of edges to a visitation order of edges in G

Note: we have no guarantee that this gives us a valid trail, since 3 subsequent vertices in L(G) might not be "traversable" in G

this must exist, as a,b,c share an endpoint in G



→ we can simply ignore that middle vertex to guarantee T is valid, and we still visit the endpoint of the middle vertex's edge in G

→ we do this for all such triplets (can explicitly modify Ham. cycle as well)

=> all subsequent 2 vertices in L(G) after "removal" of the middle vertices share an endpoint

=> which gives us a visitation order for edges in G that form a closed trail T form a closed trail T

($\Leftarrow$) We have trail $T = \{v_1 v_2 \dots v_k\}$

where $V(T)$ forms a vertex cover

→ all $e \in E(G)$ are incident to some $v \in V(T)$

→ Naively, this trail directly gives us a visitation order of edges that correspond to vertices in $L(G)$

Note: we may not necessarily visit all $v \in V(L(G)) \leftrightarrow e \in E(G)$

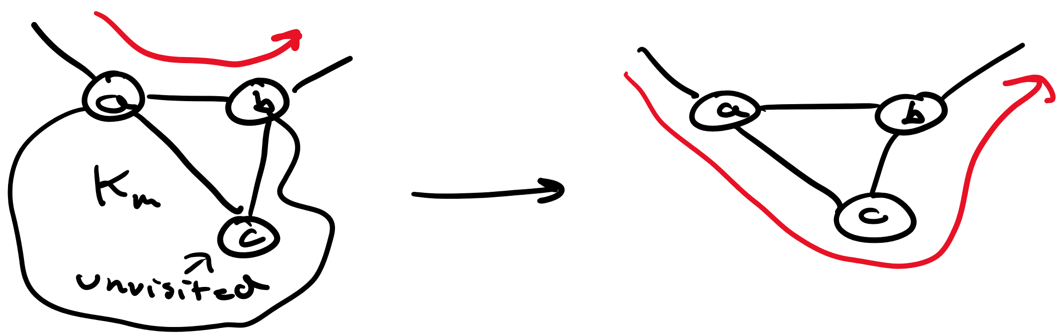
Note x2: our trail T may make repeated visits to certain edges

→ modify T to be some minimal trail T' that visits the same set of vertices

→ If there is some $v \in V(L(G))$ that we don't visit

→ this v is within a clique containing at least 2 vertices we do visit

→ we can modify our visitation

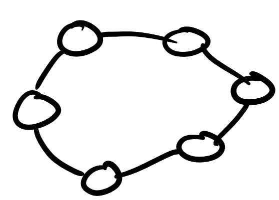


→ we can do this for any such vertex or vertices

⇒ combined with our minimality of T' and its edges → vertices of $L(G)$, we have a closed path containing all $v \in V(L(G))$

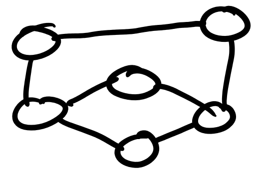
→ aka a Hamiltonian Cycle □

(7) Hamiltonian: C_6

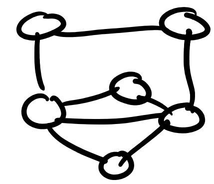


closure of $C_6 = C_6 \neq K_6$

non-Hamiltonian:



G



$C(G) \neq K_6$

(8) $\# K_3 =$ (# of ways to select 3 verts) $*$ (prob. all 3 verts are connected)

$\# K_3 = \binom{n}{3}(p^3)$ ← edge existence is independent

$\# K_3 = \binom{100}{3}(0.1^3) = \boxed{161.7 \text{ triangles}}$